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Finite time blow-up for an inhomogeneous strongly damped fourth-order wave equation with nonlinear memory

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Abstract

This paper investigates the blow-up phenomena for an inhomogeneous, strongly damped fourth-order wave equation featuring a nonlinear memory term

$$u_{tt}(t,x) - \triangle^2 u(t,x) - \triangle u_t(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} |u(s,x)|^p ds + \omega(x)$$

with initial conditions $\big(u(0,x), u_t(0,x)\big) = (u_0, u_1)$ in $\mathbb{R}^N$, where $N \geq 1, p > 1, \alpha \in (0,1)$, $u_i \in L^1_{loc}(\mathbb{R}^N)$(for $i = 1,2$) and $\omega(x) \not\equiv 0$. By employing a special class of test functions, fractional calculus techniques, and nonlinear inequality methods, we prove that provided $\omega(x) \in L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}} \omega(x) dx > 0$, the problem admits no global weak solution for any $p > 1$.

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1 Introduction

Wave equations are fundamental in modeling diverse propagation phenomena across physics, engineering, and biology [3, 7, 8]. While second-order models have been extensively studied, there is growing interest in higher-order wave equations for describing more complex systems, such as the vibrations of plates (modeled by the $\Delta^2 u$ term[11]) and materials with significant internal micro damping (modeled by the Δu_t term[1, 12]). The interplay between such higher-order dissipative mechanisms and nonlocal nonlinearities, such as memory effects, presents a rich and challenging mathematical landscape.

In this work, we analyze the following inhomogeneous, strongly damped fourth-order wave equation

$$u_{tt} - \Delta^2 u - \Delta u_t = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} |u(s,x)|^p ds + \omega(x), (t,x) \in (0,T) \times \mathbb{R}^N, \qquad (1.1)$$

subject to the initial condition:

$$\big(u(0,x), u_t(0,x)\big) = (u_0, u_1). \qquad (1.2)$$

Here, the integral term on the right-hand side represents a nonlinear memory effect with a fractional-order kernel, reflecting the influence of the historical state of the system on its current dynamics. The space-dependent inhomogeneous term $\omega(x)$ can model external forcing or a background potential.

The literature on wave equations with memory and damping is extensive. For the second-order case, the semilinear damped wave equation:

$$u_{tt} - \Delta u + u_t = |u|^p$$

has been thoroughly investigated. It is well-known that the critical Fujita exponent $1 + \frac{2}{N}$ plays a pivotal role in determining the global existence and blow-up of solutions [2, 10]. When an inhomogeneous term $\omega(x)$ is added, the critical exponent shifts, ie p to ∞ for $N = 1, 2$ and to $1 + \frac{2}{N-2}$ for $N \geq 3$ [4]. For equations with memory and inhomogeneous, Jleli et al [5] studied

$$u_{tt} - \Delta u + u_t = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} |u(s,x)|^p ds + \omega(x)$$

and established critical exponents $p > 1$. However, the interplay of a fourth-order operator, strong damping, nonlinear memory, and an inhomogeneous source remains largely unexplored.

Our main contribution is to show that if the inhomogeneous term $\omega(x)$ satisfies $\int \omega(x)dx > 0$, then no global solutions exist for any $p > 1$. This phenomenon is similar to that observed [5] for the second-order case, but our analysis must contend with the additional complexities of the fourth-order setting.

Theorem 1.1 If $u_0,\ u_1 \in C_0(\mathbb{R}^N)$ and $\omega(x) \in L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} \omega(x)dx > 0$, then problem (1.1)-(1.2) admits no global weak solution for all $p > 1$.

Remark 1.1 Let u be a weak solution of problem (1.1)-(1.2). We call u a blow-up in finite time if the maximal existence time T is finite and $u \to +\infty$ at $t \to T$.

The paper is organized as follows. In Section 2, we recall some preliminaries on fractional integrals and define the concept of a global weak solution. Section 3 is devoted to the proof of Theorem 1.1, which relies on a careful choice of test functions and a priori estimates.

2 Preliminaries

We need some properties from fractional calculus. For more details, we refer the reader to ([6]).

Let $0 < \alpha < 1$, $u \in C^\infty([0,T], \mathbb{R}^N)$ and $T > 0$. The left Riemann-Liouville fractional integrals of order α of u and the right Riemann-Liouville fractional integrals of order α of u is defined as

$$(I^\alpha_{0+} u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \qquad 0 \leq t \leq T$$

and

$$(I^\alpha_{T-} u)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} u(s) ds, \qquad 0 \leq t \leq T.$$

From Kilbas-rivastava-rjillo([6]), for $\alpha > 0$, $\varphi(t) \in L_p(0,T)$ and $u(t) \in L_p(0,T)$, one has

$$\int_0^T \varphi(t)(I^\alpha_{0^+}u)(t)dt = \int_0^T u(t)(I^\alpha_{T^-}\varphi)(t)dt.$$

Based on the introduction of the fractional integral mention above, we can define global weak solutions to (1.1) and (1.2) as follows.

Definition 2.1 Let $N \geq 1$, u_0, $u_1 \in C_0(\mathbb{R}^N)$ and $\omega(x) \in L^1(\mathbb{R}^N)$. We say $u(x,t) \in L^p\big(L^\infty_{loc}(\mathbb{R}^N),(0,T)\big)$ to be a weak solution of problem (1.1)-(1.2), if the equality

$$\int_0^T\int_{\mathbb{R}^N}(I^{1-\alpha}_{T^-}\varphi)|u|^p dxdt + \int_0^T\int_{\mathbb{R}^N}\omega\varphi dxdt + \int_{\mathbb{R}^N}\Big(u_1\varphi(0,x) - u_0\varphi_t(0,x) - u_0\Delta\varphi(0,x)\Big)dx$$
$$= \int_0^T\int_{\mathbb{R}^N}u\varphi_{tt}dxdt + \int_0^T\int_{\mathbb{R}^N}u\Delta\varphi_t dxdt - \int_0^T\int_{\mathbb{R}^N}u\Delta^2\varphi dxdt \tag{2.1}$$

holds for all $T > 0$ and $\varphi \in C^{2,4}_{t,x}\big([0,T],\mathbb{R}^N\big)$ with $supp_x\ \varphi \subset\subset \mathbb{R}^N$, $\varphi(T,\cdot) = 0$ and $\varphi_t(T,\cdot) = 0$.

Lemma 2.1([6]). If $\alpha > 0$ and $\beta > 0$, then

$$\big(I^\alpha_{T^-}(T-t)^{\beta-1}\big)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(T-x)^{\beta+\alpha-1}$$

Lemma 2.2 ([9]). Let $\omega(x) \in L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\omega(x)dx > 0$. Then there exists a test function $0 \leq \phi \leq 1$ such that $\int_{\mathbb{R}^N}\omega(x)\phi dx > 0$.

3 Proof of Theorem 1.1

3.1 Proof Outline

We proceed by contradiction, assuming that a global weak solution u exists. The core idea is to construct a test function $\varphi(t,x) = \eta(t)\mu(x)$ that localizes the problem in both space and time. Substituting φ into (2.1) and applying Young's inequality, we derive a key estimate. After carefully estimating the resulting terms, we obtain an inequality that, in the limit $T \to \infty$, contradicts the positivity condition $\int_{\mathbb{R}^N}\omega(x)\mu(x)dx > 0$.

3.2 Proof Details

Proof: The proof by contradiction. Assume that $u(x,t)$ is a global weak solution to the problem (1.1)-(1.2). Inspired by [5], we choose the test function following

$$\varphi(t,x) = \eta(t)\mu(x),$$

where

$$\eta(t) = T^{-\lambda}(T-t)^\lambda, \lambda > \frac{p+1-\alpha}{p-1}, t \in [0,T], T \in (0,\infty);$$
$$\mu(x) = \phi\left(\frac{|x|^2}{R^2}\right), R \gg 1, x \in \mathbb{R}^N.$$

Here, $\phi(z)$ is a smooth cut-off function satisfying:

(i)$|\phi'(z)| \leq C|\phi(z)|$, $|\phi''(z)| \leq C|\phi(z)|$, $|\phi'''(z)| \leq C|\phi(z)|$, $|\phi^{(4)}(z)| \leq C|\phi(z)|$;

(ii)$\phi(z) \in C^4_0(\mathbb{R}_+)$ is non-increasing function with $\phi(z) \equiv 1$ if $z \in [0,1]$ and $\phi(z) \equiv 0$ if $z \in [2,\infty)$.

Then, it follows from (2.1) that

$$\int_0^T\int_{\mathbb{R}^N}(I_{T^-}^{1-\alpha}\varphi)|u|^p dxdt+\int_0^T\int_{\mathbb{R}^N}\omega\varphi dxdt+\int_{\mathbb{R}^N}\Big(u_1\varphi(0,x)-u_0\varphi_t(0,x)-u_0\Delta\varphi(0,x)\Big)dx$$
$$\le\int_0^T\int_{\mathbb{R}^N}|u||\varphi_{tt}|dxdt+\int_0^T\int_{\mathbb{R}^N}|u||\Delta\varphi_t|dxdt+\int_0^T\int_{\mathbb{R}^N}|u||\Delta^2\varphi|dxdt. \tag{3.1}$$

Using the Young inequality with $\varepsilon=\frac{p}{3}$ to (3.1), we obtain

$$\int_0^T\int_{\mathbb{R}^N}|u||\varphi_{tt}|dxdt\le\frac{1}{3}\int_0^T\int_{\mathbb{R}^N}|u|^p(I_{T^-}^{1-\alpha}\varphi)dxdt$$
$$+\frac{p-1}{p}\left(\frac{p}{3}\right)^{-\frac{1}{p-1}}\underbrace{\int_0^T\int_{\mathbb{R}^N}(I_{T^-}^{1-\alpha}\varphi)^{-\frac{1}{p-1}}|\varphi_{tt}|^{\frac{p}{p-1}}dxdt}_{J(\varphi)} \tag{3.2}$$

$$\int_0^T\int_{\mathbb{R}^N}|u||\Delta\varphi_t|dxdt\le\frac{1}{3}\int_0^T\int_{\mathbb{R}^N}|u|^p(I_{T^-}^{1-\alpha}\varphi)dxdt$$
$$+\frac{p-1}{p}\left(\frac{p}{3}\right)^{-\frac{1}{p-1}}\underbrace{\int_0^T\int_{\mathbb{R}^N}(I_{T^-}^{1-\alpha}\varphi)^{-\frac{1}{p-1}}|\Delta\varphi_t|^{\frac{p}{p-1}}dxdt}_{K(\varphi)} \tag{3.3}$$

and

$$\int_0^T\int_{\mathbb{R}^N}|u||\Delta^2\varphi|dxdt\le\frac{1}{3}\int_0^T\int_{\mathbb{R}^N}|u|^p(I_{T^-}^{1-\alpha}\varphi)dxdt$$
$$+\frac{p-1}{p}\left(\frac{p}{3}\right)^{-\frac{1}{p-1}}\underbrace{\int_0^T\int_{\mathbb{R}^N}(I_{T^-}^{1-\alpha}\varphi)^{-\frac{1}{p-1}}|\Delta^2\varphi|^{\frac{p}{p-1}}dxdt}_{L(\varphi)}. \tag{3.4}$$

Inserting (3.2)-(3.4) into (3.1) gives

$$\int_0^T\int_{\mathbb{R}^N}\omega\varphi dxdt+\int_{\mathbb{R}^N}\big(\varphi(0,x)u_1-\varphi_t(0,x)u_0\big)dx\le C(p)\Big(J(\varphi)+K(\varphi)+L(\varphi)\Big), \tag{3.5}$$

where $C(p)=\frac{p-1}{p}\left(\frac{p}{3}\right)^{-\frac{1}{p-1}}$.

By the definition of the test function $\varphi(t,x)$ and simple calculation, we easily get

$$\int_0^T\int_{\mathbb{R}^N}\omega(x)\varphi dxdt=C(\lambda)T\int_{\mathbb{R}^N}\omega\mu dx \tag{3.6}$$

and

$$\int_{\mathbb{R}^N}\Big(u_1\varphi(0,x)-u_0\varphi_t(0,x)-u_0\Delta\varphi(0,x)\Big)dx=\int_{\mathbb{R}^N}\Big(u_1\mu(x)+\frac{\lambda u_0}{T}\mu(x)-u_0\Delta\mu(x)\Big)dx. \tag{3.7}$$

Plugging (3.6) and (3.7) into (3.5) gives

$$C(\lambda)T\int_{\mathbb{R}^N}\omega\mu dx+\int_{\mathbb{R}^N}\Big(u_1\mu(x)+\frac{\lambda u_0}{T}\mu(x)-u_0\Delta\mu(x)\Big)dx$$
$$\le C(p)\Big(J(\varphi)+K(\varphi)+L(\varphi)\Big). \tag{3.8}$$

Next, we estimate the integrals $J(\varphi)$, $K(\varphi)$ and $L(\varphi)$. These estimates constitute the technical core of the proof.

Lemma 3.1 (Estimation of $J(\varphi)$) .

We have the estimate

$$J(\varphi) \leq C(\lambda)T^{-\frac{2p+1-\alpha}{p-1}+1}R^N.$$

Proof: By the definition of the test function $\varphi(t,x)$ and simple calculation, it derives that

$$\begin{aligned}
J(\varphi) =& \int_0^T \int_{\mathbb{R}^N} (I_{T^-}^{1-\alpha}\eta(t)\mu(x))^{-\frac{1}{p-1}} |\eta''(t)\mu(x)|^{\frac{p}{p-1}} dxdt \\
\leq& C(\lambda)T^{-\lambda} \underbrace{\int_0^T \left(I_{T^-}^{1-\alpha}(T-t)^\lambda\right)^{-\frac{1}{p-1}} |T-t|^{\frac{\lambda p-2p}{p-1}} dt}_{J_{11}} \underbrace{\int_{\mathbb{R}^N} |\mu(x)| dx}_{J_{12}}.
\end{aligned}$$

Noting that by Lemma 2.1, it derives that

$$\begin{aligned}
J_{11} =& \int_0^T \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+2-\alpha)}(T-t)^{\lambda+1-\alpha}\right)^{-\frac{1}{p-1}} |T-t|^{\frac{\lambda p-2p}{p-1}} dt \\
\leq& C(\lambda) \int_0^T |T-t|^{\lambda-\frac{2p+1-\alpha}{p-1}} dt \\
\leq& C(\lambda)T^{\lambda-\frac{2p+1-\alpha}{p-1}+1}
\end{aligned}$$

and by the cut-off function $\phi(z)$, it derives that

$$\begin{aligned}
J_{12} =& \int_{\mathbb{R}^N} |\phi\left(\frac{|x|^2}{R^2}\right)| dx \\
\leq& \int_{0\leq|x|\leq R} |\phi\left(\frac{|x|^2}{R^2}\right)| dx + \int_{R\leq|x|\leq\sqrt{2}R} |\phi\left(\frac{|x|^2}{R^2}\right)| dx \\
\leq& CR^N.
\end{aligned}$$

Combining these estimates, we obtained the desired results.

Lemma 3.2 (Estimation of $K(\varphi)$) .

We have the estimate

$$K(\varphi) \leq C(\lambda)T^{-\frac{p+1-\alpha}{p-1}+1}R^{N-\frac{2p}{p-1}}.$$

Proof: Similar to the estimation of $J(\varphi)$, we have

$$\begin{aligned}
K(\varphi) =& \int_0^T \int_{\mathbb{R}^N} (I_{T^-}^{1-\alpha}\eta(t)\mu(x))^{-\frac{1}{p-1}} |\eta'(t)\Delta\mu(x)|^{\frac{p}{p-1}} dxdt \\
\leq& C(\lambda)T^{-\lambda} \underbrace{\int_0^T \left(I_{T^-}^{1-\alpha}(T-t)^\lambda\right)^{-\frac{1}{p-1}} |(T-t)|^{\frac{\lambda p-p}{p-1}} dt}_{K_{21}} \underbrace{\int_{\mathbb{R}^N} |\mu(x)|^{-\frac{1}{p-1}} |\Delta\mu(x)|^{\frac{p}{p-1}} dx}_{K_{22}}.
\end{aligned}$$

Noting that by Lemma 2.1, it derives that

$$\begin{aligned}
K_{21} =& \int_0^T \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+2-\alpha)}(T-t)^{\lambda+1-\alpha}\right)^{-\frac{1}{p-1}} |T-t|^{\frac{\lambda p-p}{p-1}} dt \\
\leq& C(\lambda) \int_0^T |T-t|^{\lambda-\frac{p+1-\alpha}{p-1}} dt \\
\leq& C(\lambda)T^{\lambda-\frac{p+1-\alpha}{p-1}+1}.
\end{aligned}$$

Combining the definition of the function $\mu(x)$ with the condition **(i)**, we have

$$\begin{aligned}
|\Delta\mu(x)| =& \left|\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)\left(\frac{\partial\phi\left(\frac{r^2}{R^2}\right)}{\partial x_1}, \frac{\partial\phi\left(\frac{r^2}{R^2}\right)}{\partial x_2}, \cdots, \frac{\partial\phi\left(\frac{r^2}{R^2}\right)}{\partial x_n}\right)\right| \\
=& \left|\left(\frac{\partial\left(\frac{2x_1}{R^2}\phi'\right)}{\partial x_1} + \frac{\partial\left(\frac{2x_2}{R^2}\phi'\right)}{\partial x_2} + \cdots + \frac{\partial\left(\frac{2x_n}{R^2}\phi'\right)}{\partial x_n}\right)\right| \\
=& \left|\frac{2}{R^2}\phi' + \frac{2x_1}{R^2}\frac{2x_1}{R^2}\phi'' + \frac{2}{R^2}\phi' + \frac{2x_2}{R^2}\frac{2x_2}{R^2}\phi'' + \cdots + \frac{2}{R^2}\phi' + \frac{2x_n}{R^2}\frac{2x_n}{R^2}\phi''\right| \\
\leq& \frac{2N}{R^2}|\phi'| + \frac{4(x_1^2 + x_2^2 + \cdots + x_n^2)}{R^4}|\phi''| \\
\leq& \frac{C}{R^2}|\phi|,
\end{aligned}$$

where $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$. Therefore, we have

$$\begin{aligned}
K_{22} \leq& CR^{-\frac{2p}{p-1}} \int_{\mathbb{R}^N} |\mu(x)| dx \\
\leq& CR^{N-\frac{2p}{p-1}}.
\end{aligned}$$

Based on the estimations in K_{11} and K_{22}, we obtain the desired result.

Lemma 3.3 (Estimation of $L(\varphi)$) .

We have the estimate

$$L(\varphi) \leq C(\lambda)T^{-\frac{1-\alpha}{p-1}+1}R^{N-\frac{4p}{p-1}}.$$

Proof: Similar to the estimation of $J(\varphi)$ and $K(\varphi)$, we have

$$\begin{aligned}
L(\varphi) =& \int_0^T \int_{\mathbb{R}^N} (I_{T^-}^{1-\alpha}\eta(t)\mu(x))^{-\frac{1}{p-1}} |\eta(t)\Delta^2\mu(x)|^{\frac{p}{p-1}} dx dt \\
\leq& T^{-\lambda} \underbrace{\int_0^T \left(I_{T^-}^{1-\alpha}(T-t)^\lambda\right)^{-\frac{1}{p-1}} |(T-t)|^{\frac{\lambda p}{p-1}} dt}_{L_{31}} \underbrace{\int_{\mathbb{R}^N} |\mu(x)|^{-\frac{1}{p-1}} |\Delta^2\mu(x)|^{\frac{p}{p-1}} dx}_{L_{32}}.
\end{aligned}$$

Noting that by Lemma 2.1, it derives that

$$\begin{aligned}
L_{31} =& \int_0^T \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+2-\alpha)}(T-t)^{\lambda+1-\alpha}\right)^{-\frac{1}{p-1}} |T-t|^{\frac{\lambda p}{p-1}} dt \\
\leq& C(\lambda) \int_0^T |T-t|^{\lambda-\frac{1-\alpha}{p-1}} dt \\
\leq& C(\lambda) T^{\lambda-\frac{1-\alpha}{p-1}+1}.
\end{aligned}$$

According to the estimate of $\Delta\mu(x)$ above, we have

$$|\Delta^2\mu(x)| = |\Delta(\Delta\mu(x))| \leq \frac{2N}{R^2}|\Delta\mu'(x)| + \frac{4}{R^4}|\Delta(r^2\mu''(x))|.$$

Once again combining the definition of the function $\mu(x)$ with the condition **(i)**, we have

$$\begin{aligned}
|\Delta\mu'(x)| =&\left|\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots,\frac{\partial}{\partial x_n}\right)\left(\frac{\partial\phi'\left(\frac{r^2}{R^2}\right)}{\partial x_1},\frac{\partial\phi'\left(\frac{r^2}{R^2}\right)}{\partial x_2},\cdots,\frac{\partial\phi'\left(\frac{r^2}{R^2}\right)}{\partial x_n}\right)\right|\\
=&\left|\left(\frac{\partial\left(\frac{2x_1}{R^2}\phi''\right)}{\partial x_1}+\frac{\partial\left(\frac{2x_2}{R^2}\phi''\right)}{\partial x_2}+\cdots+\frac{\partial\left(\frac{2x_n}{R^2}\phi''\right)}{\partial x_n}\right)\right|\\
=&\left|\frac{2}{R^2}\phi''+\frac{2x_1}{R^2}\frac{2x_1}{R^2}\phi'''+\frac{2}{R^2}\phi''+\frac{2x_2}{R^2}\frac{2x_2}{R^2}\phi'''+\cdots+\frac{2}{R^2}\phi''+\frac{2x_n}{R^2}\frac{2x_n}{R^2}\phi'''\right|\\
\le&\frac{2N}{R^2}|\phi''|+\frac{4(x_1^2+x_2^2+\cdots+x_n^2)}{R^4}|\phi'''|\\
\le&\frac{2N}{R^2}|\phi''|+\frac{4r^2}{R^4}|\phi'''|
\end{aligned}$$

and

$$\begin{aligned}
|\Delta(r^2\mu''(x))| =&\left|\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots,\frac{\partial}{\partial x_n}\right)\left(\frac{\partial r^2\phi''\left(\frac{r^2}{R^2}\right)}{\partial x_1},\frac{\partial r^2\phi''\left(\frac{r^2}{R^2}\right)}{\partial x_2},\cdots,\frac{\partial r^2\phi''\left(\frac{r^2}{R^2}\right)}{\partial x_n}\right)\right|\\
=&\left|\left(\frac{\partial\left(2x_1\phi''+r^2\frac{2x_1}{R^2}\phi'''\right)}{\partial x_1}+\frac{\partial\left(2x_2\phi''+r^2\frac{2x_2}{R^2}\phi'''\right)}{\partial x_2}+\cdots+\frac{\partial\left(2x_n\phi''+r^2\frac{2x_n}{R^2}\phi'''\right)}{\partial x_n}\right)\right|\\
=&\left|2\phi''+2x_1\frac{2x_1}{R^2}\phi'''+2x_1\frac{2x_1}{R^2}\phi'''+r^2\left(\frac{2}{R^2}\phi'''+\frac{2x_1}{R^2}\frac{2x_1}{R^2}\phi^{(4)}\right)\right.\\
&+2\phi''+2x_2\frac{2x_2}{R^2}\phi'''+2x_2\frac{2x_2}{R^2}\phi'''+r^2\left(\frac{2}{R^2}\phi'''+\frac{2x_2}{R^2}\frac{2x_2}{R^2}\phi^{(4)}\right)\\
&\left.+\cdots+2\phi''+2x_n\frac{2x_n}{R^2}\phi'''+2x_n\frac{2x_n}{R^2}\phi'''+r^2\left(\frac{2}{R^2}\phi'''+\frac{2x_n}{R^2}\frac{2x_n}{R^2}\phi^{(4)}\right)\right|\\
\le&2N|\phi''|+\frac{4(x_1^2+x_2^2+\cdots+x_n^2)}{R^2}|\phi'''|+\frac{4(x_1^2+x_2^2+\cdots+x_n^2)}{R^2}|\phi'''|\\
&+r^2\frac{2N}{R^2}|\phi'''|+r^2\frac{4(x_1^2+x_2^2+\cdots+x_n^2)}{R^4}|\phi^{(4)}|\\
=&2N|\phi''|+\frac{2N+8}{R^2}r^2|\phi'''|+\frac{4r^4}{R^4}|\phi^{(4)}|.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\Delta^2\mu(x)| \le&\frac{2N}{R^2}\left(\frac{2N}{R^2}|\phi''|+\frac{4r^2}{R^4}|\phi'''|\right)+\frac{4}{R^4}\left(2N|\phi''|+\frac{2N+8}{R^2}r^2|\phi'''|+\frac{4r^4}{R^4}|\phi^{(4)}|\right)\\
=&\frac{4N^2+8N}{R^4}|\phi''(x)|+\frac{16N+32}{R^6}r^2|\phi'''|+\frac{16}{R^8}r^4|\phi^{(4)}|\\
\le&\frac{C}{R^4}|\phi|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
L_{32} \le&CR^{-\frac{4p}{p-1}}\int_{\mathbb{R}^N}|\mu(x)|dx\\
\le&CR^{N-\frac{4p}{p-1}}.
\end{aligned}$$

Based on the above estimates, we obtained the desired results.

Now we substitute the estimates of Lemmas 3.1-3.3 into (3.8) gives

$$
\begin{aligned}
&C(\lambda)T\int_{\mathbb{R}^N}\omega\mu dx+\int_{\mathbb{R}^N}\left(u_1\mu(x)+\frac{\lambda u_0}{T}\mu(x)-u_0\Delta\mu(x)\right)dx\\
\le& C(p)\left(C(\lambda)T^{-\frac{2p+1-\alpha}{p-1}+1}R^N+C(\lambda)T^{-\frac{p+1-\alpha}{p-1}+1}R^{N-\frac{2p}{p-1}}+C(\lambda)T^{-\frac{1-\alpha}{p-1}+1}R^{N-\frac{4p}{p-1}}\right). \qquad (3.9)
\end{aligned}
$$

By rearranging the items, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\omega\mu dx+C(\lambda)T^{-1}\int_{\mathbb{R}^N}\left(u_1\mu(x)+\frac{\lambda u_0}{T}\mu(x)-u_0\Delta\mu(x)\right)dx\\
&\qquad\le C(p,\lambda)(T^{-\frac{2p+1-\alpha}{p-1}}R^N+T^{-\frac{p+1-\alpha}{p-1}}R^{N-\frac{2p}{p-1}}+T^{-\frac{1-\alpha}{p-1}}R^{N-\frac{4p}{p-1}}).
\end{aligned}
$$

Finally, for any given real number $R\gg 1$ and taking the limits $T\to\infty$ in the last inequality gives

$$
\int_{\mathbb{R}^N}\omega(x)\mu dx\le 0,
$$

by Lemma 2.2, which contradicts $\int_{\mathbb{R}^N}\omega\mu dx>0$. □

Declarations

Author's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Data Availability Statement

Not applicable.

Competing Interests

The authors declare that they have no competing interests.

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