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Existence and stability of periodic solutions in a multi-species GA predator-prey system with multiple delays on time scales

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Abstract

This paper studies the existence and stability of periodic solutions for a multi-species Gilpin-Ayala (GA) predator-prey system with multiple delays on time scales. By applying Mawhin's coincidence degree theory and inequality techniques, the existence of periodic solutions for the system is proved. On this basis, the stability of the periodic solutions is established by using Lyapunov stability theory. Finally, a numerical example is provided to verify the validity of the obtained conclusions.

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1 Introduction

This paper considers the following multi-species GA predator-prey system with multiple delays on time scales:

$$\mathcal{X}_p^{\Delta}(\tau) = r_p(\tau) - c_{p,1}(\tau)e^{\theta_p \mathcal{X}_p(\tau)} - \sum_{i=2}^{n+1} c_{p,i}(\tau) \int_{-\infty}^{0} k_{p,i-1}(s) e^{\mathcal{X}_p(\tau+s)} \Delta s - \phi_p(\tau) e^{-\mathcal{X}_p(\tau)}, \quad (1.1)$$

where $\tau \in \mathbb{T}, p = 1, 2 \ldots m$, Δ is the delta (or Hilger) derivative. $\mathbb{T}$ is a time scale with $\inf\{I_{-\infty}\} = -\infty$ and $\sup\{I_{-\infty}\} = 0$, where $I_{-\infty} = (-\infty, 0] \cap \mathbb{T}$. $\mathcal{X}_p(\tau)$ denotes the population density of the species at time $\tau \in \mathbb{T}$. $r_p(\tau) > 0$ is the natural birth rate of the species. $c_{p,1}(\tau) > 0$ indicates the intraspecific competition rates of the species. $c_{p,i}(\tau) > 0$ is the interspecific competition rates of the species. $\int_{-\infty}^{0} k_{p,i-1}(s)\, e^{\mathcal{X}(\tau+s)} \Delta s$ describes the distribution delay phenomenon in the reduction of the species population density. $k_{p,i-1}(\tau) > 0$ are the kernel functions corresponding to infinite distributed delays. $\phi_p(\tau) > 0$ represents the harvesting, fishing or capture controls of prey and predator. $\theta_p > 0$ is the measurement constants of nonlinear interference within species.

The first important aspect of our work is that the model (1.1) plays a significant role in ecosystem research. In 1973, Ayala, Gilpin and Eherenfeld [2] propositionosed the following competition model through comparing the experimental results of drosophila dynamics with the theoretical results of ten competition models.

$$\begin{cases} \dfrac{\mathrm{d}\mathcal{X}(\tau)}{\mathrm{d}\tau} = r_1\mathcal{X}(\tau)\left[1-\left(\dfrac{\mathcal{X}(\tau)}{K_1}\right)^{\theta_1} - c_{12}\dfrac{\mathcal{Y}(\tau)}{K_2}\right], \\ \dfrac{\mathrm{d}\mathcal{Y}(\tau)}{\mathrm{d}\tau} = r_2\mathcal{Y}(\tau)\left[1-\left(\dfrac{\mathcal{Y}(\tau)}{K_2}\right)^{\theta_2} - c_{21}\dfrac{\mathcal{X}(\tau)}{K_1}\right], \end{cases} \tag{1.2}$$

where $\mathcal{X}(\tau)$ and $\mathcal{Y}(\tau)$ are the population densities of two competing species at time $\tau \in \mathbb{R}$, respectively. $r_1 > 0$ and $r_2 > 0$ are inherent growth ratios. $K_1 > 0$ and $K_2 > 0$ represent the maximum number of species in the environment without competition. The nonlinear interference within species is measured by positive constants θ_1 and θ_2. $c_{12} > 0$ and $c_{21} > 0$ are the measures of competition between species.

Owing to the free selection of nonlinear measurement parameters $\theta_1, \theta_2 > 0$, system (1.2) is capable of encompassing more nonlinear ecological models. For example, when $\theta_1 = \theta_2 = 1$, (1.2) becomes the following Lotka-Volterra competitive model:

$$\begin{cases} \dfrac{\mathrm{d}\mathcal{X}(\tau)}{\mathrm{d}\tau} = r_1\mathcal{X}(\tau)\left[1-\dfrac{\mathcal{X}(\tau)}{K_1} - c_{12}\dfrac{\mathcal{Y}(\tau)}{K_2}\right], \\ \dfrac{\mathrm{d}\mathcal{Y}(\tau)}{\mathrm{d}\tau} = r_2\mathcal{Y}(\tau)\left[1-\dfrac{\mathcal{Y}(\tau)}{K_2} - c_{21}\dfrac{\mathcal{X}(\tau)}{K_1}\right]. \end{cases} \tag{1.3}$$

The dynamic characteristics of the GA ecosystem model have become a focal point of scholarly research, given the model's widespread attention since its propositionosal. Zhao [11, 14, 16] investigated the GA-type system with time delays, while He et al.[8, 12, 17] carried out research on the GA-type system with impulses. In addition, Fang et al.[5, 6] also conducted studies on the GA-type system under conditions such as food limitation.

The second important aspect of this paper is that system (1.1) combines the discrete and continuous models of the GA-predation ecosystem within the same framework. The theory of calculus on time scales was initially introduced in Hilger's doctoral dissertation [9] in 1988. The objective of this theory is to integrate the difference equation and the differential equation. More calculus theories on time scales are referred to the monograph by Bohner and Peterson [3, 4]. A time scale $\mathbb{T}$ is defined as a nonempty closed subset of $\mathbb{R}$. Consequently, system (1.1) unifies a difference system and a differential system in the following manner. When $\mathbb{T} = \mathbb{N}^+$, the model (1.1) transforms into the subsequent difference system.

$$\mathcal{X}_p(\tau+1) - \mathcal{X}_p(\tau) = r_p(\tau) - c_{p,1}(\tau)e^{\theta_p\mathcal{X}_p(\tau)} - \sum_{i=2}^{n+1} c_{p,i}(\tau) \sum_{s=-\infty}^{0} k_{p,i-1}(s)e^{\mathcal{X}(\tau+s)} - \phi_p(\tau)e^{-\mathcal{X}_p(\tau)}. \tag{1.4}$$

When $\mathbb{T} = \mathbb{R}$, let $x_p(\tau) = \mathrm{e}^{\mathcal{X}_p(\tau)}$, then the following differential system is obtained from model (1.1).

$$\frac{\mathrm{d}x_\mathrm{p}(\tau)}{\mathrm{d}\tau} = x_p(\tau)\left[r_p(\tau) - c_{p,1}(\tau)[x_p(\tau)]^{\theta_p} - \sum_{i=2}^{n+1} c_{p,i}(\tau)\int_{-\infty}^{0} k_{p,i-1}(s)x_p(\tau+s)\mathrm{d}s\right] - \phi_p(\tau), \quad \tau \in \mathbb{R}. \tag{1.5}$$

Because a time scale $\mathbb{T}$ actually has more complex forms like $\mathbb{T} = \bigcup_{n=-\infty}^{+\infty}[n, n+\frac{1}{4}] \cup [n+\frac{3}{4}, n+1]$, system (1.1) encompasses certain intricate systems possessing characteristics like piecewise continuity and infinite discontinuity points. Moreover, due to the presence of distributed delay terms, system (1.1) falls within the category of functional differential equations. So, our study can make the basic theory of ordinary differential equations and functional differential equations richer, especially in the existence of solutions and the stability theory of discontinuous systems.

In addition, Model (1.1) accurately characterizes real-world ecosystems through two core dimensions: temporal delays and multi-species interactions:Temporal delays are ubiquitous in predation processes; incorporating multiple delays (e.g., lag in prey growth, predator reproduction) better captures the complex dynamic coupling between predators and prey than a single delay assumption. Multi-species interactions are a structural norm of ecological communities: interspecific competition occurs within the same trophic level (e.g., different predators competing for shared prey), while multi-trophic level relationships involve nested predation and mutualism, forming intricate food webs that enhance ecosystem resilience but also increase vulnerability to disturbances. Combined with periodic changes in abiotic factors such as climate and food availability, the system exhibits inherent complexity. Targeted interventions (e.g., regulating population densities, protecting key species) are thus needed to maintain the stability and persistence of such time-delayed multi-species ecosystems.

There is a scarcity of papers that focus on the periodic solution and its stability of the GA-ecosystem within the context of time scales. For example, the papers [11, 12, 14, 15] on GA-ecosystem almost all studied continuous differential equation models. There seem to be few discrete models. Only in Fang and Wang [6] and Zhao [13, 16] did the authors study the existence, multiplicity and stability of periodic solutions to some GA-type predator-prey systems with harvesting terms on time scales. Moreover, we draw a comparison between some earlier papers regarding the GA-ecosystem and our own research efforts. Zhao [16] has proved the existence and stability of the periodic solutions for a two-species GA-predation system with a single time delay. Divya and Abbas [1] only studied the existence of the periodic solutions of the equations with multiple time delays and didn't give the stability. Based on this, this paper presents the existence and stability of the periodic solutions for multi-species GA-predation systems with multiple time delays on time scales by using Mawhin's coincidence degree theory and inequality techniques.

The rest of the paper is organized in the following way. In Section 2, we mainly present the fundamental concepts and results related to time scales, along with Mawhin's coincidence theorem and essential lemmas. Section 3 is dedicated to deriving sufficient conditions for the existence of periodic solutions. Subsequently, in Section 4, we will utilize the Lyapunov stability theory to prove the global asymptotic stability of this periodic solution, and finally illustrate the validity of the conclusion through a numerical example.

2 Preliminaries

This section begins with a brief review of the fundamental results for calculus on time scales. For more knowledge on time scales, it is referred to [3] and [4].

A nonempty closed subset $\mathbb{T} \subset \mathbb{R}$ is called a time scale. The two jump operators forward and backward denoted by $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, respectively, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$, are defined by

$$\begin{aligned} \sigma(\tau) &= \inf\{s \in \mathbb{T} : s > \tau\}, \\ \rho(\tau) &= \sup\{s \in \mathbb{T} : s < \tau\}, \\ \mu(\tau) &= \sigma(\tau) - \tau, \end{aligned}$$

for all $\tau \in \mathbb{T}$. A point $\tau \in \mathbb{T}$ is called left-dense (right-dense) when $\tau > \inf \mathbb{T}$ and $\rho(\tau) = \tau$

($\tau < \sup \mathbb{T}$ and $\sigma(\tau) = \tau$), left-scattered (right-scattered) when $\rho(\tau) < \tau$ ($\sigma(\tau) > \tau$). If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Let $\mathbb{T}$ be a time scale, if there exists a constant $\omega > 0$ such that $\tau + \omega \in \mathbb{T}$ holds for all $\tau \in \mathbb{T}$, then $\mathbb{T}$ is called an ω-periodic time scale. Obviously, if $\mathbb{T}$ is an ω-periodic time scale, we can conclude that $\mathbb{T}$ is unbounded above. We then recall some basic concepts of calculus on time scale which are taken from Bohner and Peterson [3]. The following three definitions provide the propositionerties of functions on time scales.

Definition 2.1. A function $u : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-side limits $u(\tau^+)$ and left-side limits $u(\tau^-)$ all exist (finite) for all $\tau \in \mathbb{T}$.

Definition 2.2. A function $u : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $u : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.3. Assume $u : \mathbb{T} \to \mathbb{R}$ and $\tau \in \mathbb{T}^k$. Then $u^\Delta(\tau)$ is defined to be the number (if exists) satisfying that, for any given $\varepsilon > 0$ there exists a neighbourhood U of τ (i.e. $U = (\tau - \delta, \tau + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[u(\sigma(\tau)) - u(s)] - u^\Delta(\tau)[\sigma(\tau) - s]| < \varepsilon|\sigma(\tau) - s|, \quad \forall\, s \in U.$$

$u^\Delta(\tau)$ is called the delta (or Hilger) derivative of u at τ. The set of functions $u : \mathbb{T} \to \mathbb{R}$ that are Δ-differentiable with $u^\Delta(\tau)$ being rd-continuous, is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.

According to the above definitions, we easily know that if u is Δ-differentiable, then u is continuous, and so, u is rd-continuous, and then u is regulated. Based on the above concepts, the following lemma can be obtained, and its proof is referred to [3].

Lemma 2.4. *Let u be regulated, then there is a region D and a function F such that*

$$F^\Delta(\tau) = u(\tau), \quad \forall \tau \in D. \tag{2.1}$$

According to Lemma 2.4, we continue to give the definition of integral on time scales.

Definition 2.5. Assume $u : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as (2.1) is called a Δ-antiderivative of u. The indefinite integral of a regulated function u is defined by

$$\int u(\tau)\Delta\tau = F(\tau) + C,$$

here C is an arbitrary constant and F is a Δ-antiderivative of u. We define the definite integral by

$$\int_a^b u(s)\Delta s = F(b) - F(a), \quad \forall a, b \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $u : \mathbb{T} \to \mathbb{R}$ provided

$$F^\Delta(\tau) = u(\tau), \quad \forall \tau \in \mathbb{T}^k.$$

The following lemma states some propositionerties of integral on time scales, for its proof it is referred to [3] and [4].

Lemma 2.6. *If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $u, v \in C_{rd}(\mathbb{T}, \mathbb{R})$, then*

(i) $\int_a^b [\alpha u(\tau) + \beta v(\tau)]\Delta\tau = \alpha \int_a^b u(\tau)\Delta\tau + \beta \int_a^b v(\tau)\Delta\tau$;

(ii) $\forall a \leq \tau < b, u(\tau) \geq 0$ implies $\int_a^b u(\tau)\Delta\tau \geq 0$;

(iii) $\forall \tau \in [a, b) \triangleq \{\tau \in \mathbb{T} : a \leq \tau < b\}, |u(\tau)| \leq \nu(\tau)$ implies $|\int_a^b u(\tau)\Delta\tau| \leq \int_a^b \nu(\tau)\Delta\tau$.

The definition of the degree function and the related content of Mawhin's coincidence theorem are given below, for details it is referred to [7].

Definition 2.7. Let $g \in C^1(\Omega, \mathbb{R}^n) \bigcap C(\overline{\Omega}, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded set and $y \in \mathbb{R}^n \backslash g(\partial\Omega \cup N_g)$ where N_g is the critical set of g which is defined as $N_g = [z \in \Omega : \mathbb{J}_g(z) = 0]$, here $\mathbb{J}_g(z)$ is the Jacobian of g at z. Then the degree function $\deg[g; \Omega; y]$ is defined by

$$\deg[g; \Omega; y] = \sum_{z \in g^{-1}(y)} sgn\mathbb{J}_g(z)$$

with $\sum_{z \in \emptyset} sgn\mathbb{J}_g(z) = 0$.

In order to better understand and apply the concept of Mawhin's coincidence theorem, we present the following proposition.

Proposition 2.8. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces. If a linear operator $\mathscr{L} : Dom(\mathscr{L}) \subset \mathbb{X} \to \mathbb{X}$ satisfies $codimIm(\mathscr{L}) = dimKer(\mathscr{L}) =$ finite value and that $Im(\mathscr{L})$ is closed in $\mathbb{X}$, then it is a Fredholm operator with index 0. Then continuous projectors $\mathscr{P} : \mathbb{X} \to \mathbb{X}$ and $\mathscr{Q} : \mathbb{Y} \to \mathbb{Y}$ exist such that $Im(\mathscr{P}) = Ker(\mathscr{L})$ and $Ker(\mathscr{Q}) = Im(\mathscr{L}) = Im(I - \mathscr{Q})$. As a result, the inverse of $\mathscr{L}|Dom(\mathscr{L}) \cap Ker(\mathscr{P}) : (I - \mathscr{P})\mathbb{X} \to Im(\mathscr{L})$ exists, which is denoted by $\mathcal{K}$. Let $\mathscr{N} : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. If Ω is an open bounded subset of $\mathbb{X}$, then the mapping $\mathscr{N}$ will be called $\mathscr{L}$-compact on $\overline{\Omega}$ if $\mathcal{K}(I - \mathscr{Q})\mathscr{N} : \overline{\Omega} \to \mathbb{X}$ is compact and $\mathscr{Q}\mathscr{N}(\overline{\Omega})$ is bounded. Since $Im(\mathscr{Q})$ is isomorphic to $Ker(\mathscr{L})$, therefore, there exists an isomorphism $\mathscr{J} : Im(\mathscr{Q}) \to Ker(\mathscr{L})$.*

Based on the above proposition, the specific content of Mawhin's coincidence theorem is introduced as follows.

Lemma 2.9. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces, $\Omega \subset \mathbb{X}$ be a nonempty bounded open set. Let $\mathscr{L} : \mathbb{X} \to \mathbb{Y}$ be a zero index Fredholm operator, $\mathscr{N} : \mathbb{X} \to \mathbb{Y}$ be a $\mathscr{L}$-compact operator on $\overline{\Omega}$, $\mathscr{Q} : \mathbb{Y} \to \mathbb{Y}$ be a projection operator, and $\mathscr{J} : \mathbb{Y} \to \mathbb{Y}$ be a homotopy operator. Assume that*
(i) every solution w of $\mathscr{L}w = \lambda\mathscr{N}w$ satisfies $w \notin \partial\Omega \cap \mathrm{Dom}(\mathscr{L}), \forall\lambda \in (0, 1)$;
(ii) $\mathscr{Q}\mathscr{N}w \neq 0, \forall w \in \partial\Omega \cap Ker(\mathscr{L})$;
(iii) $deg(\mathscr{J}\mathscr{Q}\mathscr{N}w; \Omega \cap Ker(\mathscr{L}); 0) \neq 0$,
then $\mathscr{L}w = \mathscr{N}w$ has at least one solution in $\overline{\Omega} \cap Dom(\mathscr{L})$.

We have the following lemma for proving propositionerties of functions with the typical form as $h(z) = a\mathrm{e}^{(1+\theta)z} - b\mathrm{e}^z + c$. For sake of simplicity, the proof is omitted and is referred to [16].

Lemma 2.10. *Let $a, b, c, \theta > 0$ be some constants, consider the function $h(z) = a\mathrm{e}^{(1+\theta)z} - b\mathrm{e}^z + c$. Assume that $\theta a^{-\frac{1}{\theta}}(\frac{b}{1+\theta})^{\frac{1+\theta}{\theta}} > c$, then the following assertions are true:*
(i) h(z) has a unique minimum point $z_0 = \frac{1}{\theta}\ln[\frac{b}{a(1+\theta)}]$ in $(-\infty, +\infty)$, and the minimum $h(z_0) =$

$-\theta a^{-\frac{1}{\theta}}(\frac{b}{1+\theta})^{\frac{1+\theta}{\theta}} + c < 0$;

(ii) h(z) is strictly decreasing in $(-\infty, z_0]$ and increasing in $[z_0, +\infty)$, respectively;
(iii) h(z) has only two zeros z_1 and z_2 satisfying $-\infty < z_1 < z_0 < z_2 < +\infty$.

We will use the following symbols for simplicity in the rest of this paper: κ is defined as the minimum of $\{[0,+\infty)\cap\mathbb{T}\}$, I_ω is equal to $[\kappa,\kappa+\omega]\cap\mathbb{T}$, $\overline{u}$ is the supremum of $u(\tau)$ with $\tau\in I_\omega$, $\underline{u}$ is the infimum of $u(\tau)$ with $\tau\in I_\omega$, and $\hat{u}$ is equal to $\frac{1}{\omega}\int_{I_\omega}u(s)\Delta s$ which is also equal to $\frac{1}{\omega}\int_\kappa^{\kappa+\omega}u(s)\Delta s$, for $u\in C_{rd}(\mathbb{T},\mathbb{R})$ satisfying $u(\tau+\omega)=u(\tau)$ for all $\tau\in\mathbb{T}$.

3 Existence of periodic solution on time scales

This section focuses on the existence of periodic solution for system (1.1) by applying Mawhin's coincidence theory (Lemma 2.9). The main idea is to construct homotopies and utilize the propositionerties of coincidence degrees to bypass the difficulties of direct solutions and demonstrate the existence of solutions from a topological perspective. In order to obtain the existence region of the solution of system (1.1), we need to utilize the following lemma from [16].

Lemma 3.1. *Let $\mathbb{T}$ be an ω-periodic time scale. Suppose $\psi:\mathbb{T}\to\mathbb{R}$ is an ω-periodic function which is rd-continuous, then*

$$0\le\sup_{s\in I_\omega}\psi(s)-\inf_{s\in I_\omega}\psi(s)\le\frac{1}{2}\int_\kappa^{\kappa+\omega}|\psi^\Delta(s)|\Delta s.$$

Now, as stated at the beginning of this section, we can use Mawhin's coincidence theory and the lemma above to obtain the existence of periodic solutions to system (1.1). We need the following assumptions in order to establish our results.

Assumption 3.2. For a constant $\omega>0$, let $\mathbb{T}$ be an ω-periodic time scale and satisfy $\inf\{I_{-\infty}\}=-\infty$ and $\sup\{I_{-\infty}\}=0$, where $I_{-\infty}=(-\infty,0]\cap\mathbb{T}$. All references to ω in this paper are consistent with this definition.

Assumption 3.3. Suppose that $0<r_p(\tau),\phi_p(\tau),c_{p,1}(\tau),c_{p,i}(\tau)\in C_{rd}(\mathbb{R})$ are all ω-periodic, and $0<k_{p,i-1}(\tau)\in C_{rd}(\mathbb{T},\mathbb{R})$ satisfy $\int_{-\infty}^0 k_{p,i-1}(s)\Delta s<\infty$, where $p=1,2,\cdots m, i=2,3,\cdots n+1$.

Assumption 3.4. Suppose the following inequalities hold:

$$\theta_p(\underline{c_{p,1}}\mathrm{e}^{-\omega\theta_p\overline{r_p}})^{-\frac{1}{\theta_p}}\left(\frac{\overline{r_p}}{1+\theta_p}\right)^{\frac{1+\theta_p}{\theta_p}}>\underline{\phi_p},$$

where $\underline{c_{p,1}}$ is the infimum of $c_{p,1}(\tau)$ with $\tau\in I_\omega$, $\overline{r_p}$ is the supremum of $r_p(\tau)$ with $\tau\in I_\omega$, $\underline{\phi_p}$ is the infimum of $\phi_p(\tau)$ with $\tau\in I_\omega$.

Theorem 3.5. *Suppose that Assumption 1-Assumption 3 hold, then the model (1.1) has at least one ω-periodic solution $(\widetilde{\mathcal{X}}_1(\tau),\widetilde{\mathcal{X}}_2(\tau),\ldots,\widetilde{\mathcal{X}}_m(\tau))^T$ on the time scale $\mathbb{T}$ which satisfies $\alpha_{p_1}-\omega\overline{r_p}<\widetilde{\mathcal{X}}_P(\tau)<\alpha_{p_2}$, where $\alpha_{p_1}<\alpha_{p_2}$, and α_{p_1} and α_{p_2} are the only two real roots of equation $f_p(Z)=0$ defined as*

$$f_p(Z)=(\underline{c_{p,1}}\,\mathrm{e}^{-\omega\theta_p\overline{r_p}})\,\mathrm{e}^{(1+\theta_p)Z}-\overline{r_p}\,\mathrm{e}^Z+\underline{\phi_p}=0.$$

Proof. **Step 1.** In order to apply Lemma 2.9, we define

$$\mathbb{X}=\mathbb{Y}=\{(\omega_1(\tau),\ldots\omega_p(\tau),\ldots,\omega_m(\tau))^{\mathrm{T}}\in C_{rd}(\mathbb{T},\mathbb{R}^m):\omega_p(\tau+\omega)=\omega_p(\tau),\omega_p(\tau)\ge0,\forall\tau\in\mathbb{T}\},$$

with norm

$$||(\omega_1(\tau),\ldots,\omega_p(\tau),\ldots,\omega_m(\tau))^{\mathrm{T}}||=\sum_{p=1}^m\max_{\tau\in I_\omega}|\omega_p(\tau)|,$$

where $(\omega_1(\tau),\ldots,\omega_p(\tau),\ldots,\omega_m(\tau))^{\mathrm{T}} \in \mathbb{X} = \mathbb{Y}$. We can easily see that the spaces $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces endowed with this norm. Now let us define operators $\mathscr{L} : Dom(\mathscr{L}) \subset \mathbb{X} \to \mathbb{Y}$ given by

$$\mathscr{L}\begin{bmatrix}\mathcal{X}_1\\ \vdots\\ \mathcal{X}_m\end{bmatrix} = \begin{bmatrix}\mathcal{X}_1^{\Delta}\\ \vdots\\ \mathcal{X}_m^{\Delta}\end{bmatrix},$$

and $\mathscr{N}((\mathcal{X}_1,\ldots\mathcal{X}_m)^{\mathrm{T}}) : \mathbb{X} \to \mathbb{Y}$ defined by

$$\mathscr{N}((\mathcal{X},\mathcal{Y})^{\mathrm{T}}) = \begin{bmatrix}w_1(\tau)\\ \vdots\\ w_p(\tau)\\ \vdots\\ w_m(\tau)\end{bmatrix},$$

where $w_p(\tau) = r_p(\tau) - c_{p,1}(\tau)e^{\theta_p\mathcal{X}_p(\tau)} - \sum_{i=2}^{n+1} c_{p,i}(\tau)\int_{-\infty}^{0} k_{p,i-1}(s)e^{\mathcal{X}_p(\tau+s)}\Delta s - \phi_p(\tau)e^{-\mathcal{X}_p(\tau)}$. Here as $\mathscr{L}$ is a linear operator and so we can find the kernel and image of this operator which are given by

$$\mathrm{Ker}(\mathscr{L}) = \{\omega = (\mathcal{X}_1(\tau),\ldots,\mathcal{X}_m(\tau))^{\mathrm{T}} \in \mathbb{X} : (\mathcal{X}_1(\tau),\ldots,\mathcal{X}_m(\tau))^{\mathrm{T}} \equiv (D_1,\ldots,D_m)^{\mathrm{T}} \in \mathbb{R}^m\},$$

and

$$\mathrm{Im}(\mathscr{L}) = \{\omega = (\mathcal{X}_1(\tau),\ldots,\mathcal{X}_m(\tau))^{\mathrm{T}} \in \mathbb{Y} : (\hat{\mathcal{X}}_1,\ldots,\hat{\mathcal{X}}_m)^{\mathrm{T}} = (0,\ldots,0)\}.$$

Since $\mathrm{Im}(\mathscr{L})$ is closed and $dimKer(\mathscr{L}) = m = codimIm(\mathscr{L})$, then $\mathscr{L}$ is a Fredholm operator of index 0. Thus by proposition 2.8, there exist projection operators $\mathscr{P} : \mathbb{X} \to \mathbb{X}$ and $\mathscr{Q} : \mathbb{Y} \to \mathbb{Y}$ such that

$$\mathscr{P}\begin{bmatrix}\mathcal{X}_1\\ \vdots\\ \mathcal{X}_m\end{bmatrix} = \mathscr{Q}\begin{bmatrix}\mathcal{X}_1\\ \vdots\\ \mathcal{X}_m\end{bmatrix} = (\widehat{\mathcal{X}_1},\ldots,\widehat{\mathcal{X}_m})^{\mathrm{T}} = \Big(\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}\mathcal{X}_1(\tau)\Delta\tau,\ldots,\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}\mathcal{X}_m(\tau)\Delta\tau\Big)^{\mathrm{T}}.$$

Notice that both projections satisfy the propositionerty $\mathrm{Im}(\mathscr{P}) = \mathrm{Ker}(\mathscr{L})$ and $\mathrm{Ker}(\mathscr{Q}) = \mathrm{Im}(\mathscr{L})$. Then we can conclude that the inverse of $\mathscr{L} : Dom(\mathscr{L}) \cap Ker\mathscr{P} \to Im\mathscr{L}$ exists, which is denoted by $\mathcal{K} : Im\mathscr{L} \to Ker\mathscr{P} \cap Dom\mathscr{L}$, and given by

$$\mathcal{K}\begin{bmatrix}\mathcal{X}_1\\ \vdots\\ \mathcal{X}_m\end{bmatrix} = \begin{bmatrix}\dot{\mathcal{X}}_1 - \widehat{\dot{\mathcal{X}}_1}\\ \vdots\\ \dot{\mathcal{X}}_m - \widehat{\dot{\mathcal{X}}_m}\end{bmatrix},$$

where $\dot{\mathcal{X}}_p = \int_{\kappa}^{\tau}\mathcal{X}_p(s)\Delta s$. We get

$$\mathscr{Q}\mathscr{N}((\mathcal{X}_1,\ldots,\mathcal{X}_m)^{\mathrm{T}}) = \begin{bmatrix}\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} w_1(\tau)\Delta\tau\\ \vdots\\ \frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} w_m(\tau)\Delta\tau\end{bmatrix}.$$

Moreover,

$$\mathcal{K}(I-\mathscr{Q})\mathscr{N}((\mathcal{X}_1,\ldots,\mathcal{X}_m)^{\mathrm{T}}) = \begin{bmatrix}\int_{\kappa}^{\tau} w_1(s)\Delta s - \frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}\int_{\kappa}^{\tau} w_1(s)\Delta s\Delta\tau - (\tau-\kappa-\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}(\tau-\kappa)\Delta\tau)\widehat{w_1(\tau)}\\ \vdots\\ \int_{\kappa}^{\tau} w_m(s)\Delta s - \frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}\int_{\kappa}^{\tau} w_m(s)\Delta s\Delta\tau - (\tau-\kappa-\frac{1}{\omega}\int_{\kappa}^{\kappa+\omega}(\tau-\kappa)\Delta\tau)\widehat{w_m(\tau)}\end{bmatrix}.$$

Since $\mathbb{X}$ is a Banach space, for any open bounded set $\Omega \subset \mathbb{X}$, $\mathcal{K}(I-\mathscr{Q})\mathscr{N}(\overline{\Omega})$ is relatively compact by Arzela-Ascoli theorem. Moreover, $\mathscr{Q}\mathscr{N}(\overline{\Omega})$ is bounded. Thus, $\mathscr{N}$ is $\mathscr{L}$-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

Step 2. Now we are committed to finding the existence region $\Omega \subset \mathbb{X}$ of solution. Assume that the operator equation $\mathscr{L}\omega = \lambda\mathscr{N}\omega$ has an ω-periodic solution $\omega = (\mathcal{X}_1, \dots, \mathcal{X}_m)^T \in \mathbb{X}$, then we have

$$\mathcal{X}_p^{\Delta}(\tau) = \lambda[r_p(\tau) - c_{p,1}(\tau)\mathrm{e}^{\theta_p \mathcal{X}_p(\tau)} - \sum_{i=2}^{n+1} c_{p,i}(\tau)\int_{-\infty}^{0} k_{p,i-1}(s)e^{\mathcal{X}_p(\tau+s)}\Delta s - \phi_p(\tau)e^{-\mathcal{X}_p(\tau)}]. \quad (3.1)$$

Integrating both sides of (3.1) yields that

$$0 = \int_{\kappa}^{\kappa+\omega}[r_p(\tau) - c_{p,1}(\tau)\mathrm{e}^{\theta_p \mathcal{X}_p(\tau)} - \sum_{i=2}^{n+1} c_{p,i}(\tau)\int_{-\infty}^{0} k_{p,i-1}(s)e^{\mathcal{X}_p(\tau+s)}\Delta s - \phi_p(\tau)e^{-\mathcal{X}_p(\tau)}]\Delta\tau. \quad (3.2)$$

Since $\mathcal{X}_p(\tau)$ is ω-periodic, there are $\mu_{p_1}, \mu_{p_2} \in I_\omega$ such that$\mathcal{X}_p(\mu_{p_1}) = \overline{\mathcal{X}_p}$, $\mathcal{X}_p(\mu_{p_2}) = \underline{\mathcal{X}_p}$, From the first equation of (3.1) and (3.2), we get

$$\int_{\kappa}^{\kappa+\omega} |\mathcal{X}_p^{\Delta}(\tau)|\Delta\tau < 2\omega\overline{r_p}. \quad (3.3)$$

By the first equation of (3.2) and (3.3) together with Lemma 3.1, we have

$$\begin{aligned}
\omega\overline{r_p} \geq & \int_{\kappa}^{\kappa+\omega} r_p(s)\Delta s \\
= & \int_{\kappa}^{\kappa+\omega} c_{p,1}(\tau)\mathrm{e}^{\theta_p\mathcal{X}_p(\tau)}\Delta\tau + \int_{\kappa}^{\kappa+\omega}\sum_{i=2}^{n+1} c_{p,i}(\tau)\int_{-\infty}^{0} k_{p,i-1}(s)e^{\mathcal{X}_p(\tau+s)}\Delta s\Delta\tau \\
& + \int_{\kappa}^{\kappa+\omega}\phi_p(\tau)e^{-\mathcal{X}_p(\tau)}\Delta\tau \\
> & \omega\underline{c_{p,1}}\mathrm{e}^{\theta_p\mathcal{X}_p(\mu_{p_2})} + \omega\underline{\phi_p}\mathrm{e}^{-\mathcal{X}_p(\mu_{p_1})} \\
\geq & \omega\underline{c_{p,1}}\mathrm{e}^{\theta_p[\mathcal{X}_p(\mu_{p_1})-\omega\overline{r_p}]} + \omega\underline{\phi_p}\mathrm{e}^{-\mathcal{X}_p(\mu_{p_1})},
\end{aligned}$$

which implies that

$$(\underline{c_{p,1}}\mathrm{e}^{-\omega\theta_p\overline{r_p}})\mathrm{e}^{(1+\theta_p)\mathcal{X}_p(\mu_{p_1})} - \overline{r_p}\mathrm{e}^{\mathcal{X}_p(\mu_{p_1})} + \underline{\phi_p} < 0. \quad (3.4)$$

Based on Lemma 2.10, we know that $f_p(Z)$ has a unique minimum point α_{p_0} and a minimum $f_p(\alpha_{p_0})$ as follows:

$$\alpha_{p_0} = \frac{1}{\theta_p}\ln\left[\frac{\overline{r_p}}{(\underline{c_{p,1}}\,\mathrm{e}^{-\omega\theta_p\overline{r_p}})(1+\theta_p)}\right],$$

$$f_p(\alpha_{p_0}) = -\theta_p(\underline{c_{p,1}}\,\mathrm{e}^{-\omega\theta_p\overline{r_p}})^{-\frac{1}{\theta_p}}\left(\frac{\overline{r_p}}{1+\theta_p}\right)^{\frac{1+\theta_p}{\theta_p}} + \underline{\phi_p}.$$

From Assumption 3 and Lemma 2.10, we conclude that $f_p(\alpha_{p_0}) < 0$, and there are only two constants α_{p_1} and α_{p_2} such that

$$\alpha_{p_1} < \alpha_{p_0} < \alpha_{p_2}, \quad f_p(\alpha_{p_1}) = f_p(\alpha_{p_2}) = 0. \quad (3.5)$$

By (3.5) and Lemma 2.10, we find that the solution of the inequality (3.4) as

$$\alpha_{p_1} < \mathcal{X}_p(\mu_{p_1}) < \alpha_{p_2}. \quad (3.6)$$

Using Lemma 3.1 and (3.6), we obtain

$$\alpha_{p_1} - \omega\overline{r_p} < \mathcal{X}_p(\mu_{p_2}) \leq \mathcal{X}_p(\mu_{p_1}) < \alpha_{p_2}, \quad p = 1, 2, \cdots, m. \tag{3.7}$$

Based on (3.7), we have

$$\Omega = \left\{(\mathcal{X}_1(\tau), \mathcal{X}_2(\tau), \cdots, \mathcal{X}_m(\tau))^T : \alpha_{p_1} - \omega\overline{r_p} < \mathcal{X}_p(\tau) < \alpha_{p_2}, p = 1, 2, \cdots, m\right\}.$$

Obviously, $\Omega \subset \mathbb{X}$ is a bounded open subset satisfying the condition (i) in Lemma 2.9.

Step 3. Next, it is necessary to verify that condition (ii) of Lemma 2.9 is true, namely, for $\omega \in \partial\Omega \cap Ker(\mathscr{L}) = \partial\Omega \cap \mathbb{R}^m$, $\mathscr{Q}\mathscr{N}\omega \neq (0,0)$. If it is not true, then for $\omega \in \partial\Omega \cap Ker(\mathscr{L}) = \partial\Omega \cap \mathbb{R}^m$, there is a constant vector $\omega^* = (\mu_1{}^*, \cdots, \mu_m{}^*)$ with $\omega^* \in \partial\Omega \cap \mathbb{R}^m$ such that

$$\int_{\kappa}^{\kappa+\omega} \left[r_p(\tau) - c_{p,1}(\tau)\mathrm{e}^{\theta_p u_p^*} - \sum_{i=2}^{n+1} c_{p,i}(\tau)\mathrm{e}^{\mu_p^*} \int_{-\infty}^{0} k_{p,i-1}(s)\Delta s - \phi_p(\tau)\mathrm{e}^{-\mu_p{}^*} \right] \Delta\tau = 0. \tag{3.8}$$

The discussion for (3.8) similar to (3.7) yields $\omega^* = (\mu_1{}^*, \cdots, \mu_m{}^*) \in \Omega \cap \mathbb{R}^m$, which contradicts with $\omega \in \partial\Omega \cap \mathbb{R}^m$. Thus condition (ii) in Lemma 2.9 is true.

Let $\mathscr{J} = Id$ be the identity mapping. In oder to calculate the degree, we define a homotopy as follows:

$$\mathcal{H}(\omega, y) = (y)\mathscr{Q}\mathscr{N}\omega + (1-y)M \quad for \quad y \in (0,1),$$

where

$$M = \begin{bmatrix} \frac{1}{2}(\alpha_{11} - \omega\overline{r_1} + \alpha_{12}) - \mathcal{X}_1 \\ \vdots \\ \frac{1}{2}(\alpha_{p_1} - \omega\overline{r_p} + \alpha_{p_2}) - \mathcal{X}_p \\ \vdots \\ \frac{1}{2}(\alpha_{m_1} - \omega\overline{r_m} + \alpha_{m_2}) - \mathcal{X}_m \end{bmatrix}.$$

For any $\mathcal{X} \in \partial\Omega \cap Ker(\mathscr{L})$, we have $\mathcal{H}(\omega, y) \neq 0$. By homotopic invariance of topological degree, we get

$$\begin{aligned} deg\{\mathscr{J}\mathscr{Q}\mathscr{N}\omega; \Omega \cap Ker\mathscr{L}; 0\} &= \deg\{\mathscr{Q}\mathscr{N}\omega; \Omega \cap Ker\mathscr{L}; 0\} \\ &= \deg\{M; \Omega \cap Ker\mathcal{L}; 0\} \\ &\neq 0. \end{aligned}$$

Thus all the assumptions of Lemma 2.9 are justified. Consequently, system (1.1) has at least one ω-periodic solution $(\widetilde{\mathcal{X}}_1(\tau), \cdots, \widetilde{\mathcal{X}}_p(\tau), \cdots, \widetilde{\mathcal{X}}_m(\tau))^{\mathrm{T}}$. The proof is completed. □

4 Global asymptotic stability

In this section, we concentrate on the stability of the periodic solution of model (1.1). According to Definition 1.6 in Lakshmikantham and Vatsala [10], we define the following generalised derivative (or Dini derivative).

Definition 4.1. For each $\tau \in \mathbb{T}$, let U be a neighbourhood of τ. Then, for $V \in C_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+)$, define the Dini derivative $D^+V^{\Delta}(\tau, x(\tau))$, that is, for $\varepsilon > 0$, there is a right neighbourhood $U_\varepsilon \cap U$ of τ such that

$$\frac{V(\sigma(\tau), x(\sigma(\tau))) - V(s, x(s))}{\sigma(\tau) - s} < D^+V^{\Delta}(\tau, x(\tau)) + \varepsilon, \quad \forall s \in U_\varepsilon, s > \tau.$$

If τ is right-scattered and $V(\tau, x(\tau))$ is continuous at τ, this can be reduced to

$$D^+V^{\Delta}(\tau, x(\tau)) = \frac{V(\sigma(\tau), x(\sigma(\tau))) - V(\tau, x(\tau))}{\sigma(\tau) - \tau}.$$

By Theorem 3.5, we conclude that system (1.1) has at least one ω-periodic solution $(\widetilde{\mathcal{X}}_1(\tau), \cdots, \widetilde{\mathcal{X}}_p(\tau),$
$\cdots, \widetilde{\mathcal{X}}_m(\tau))^{\mathrm{T}} \in \Omega$. Let $u_p(\tau) = \mathrm{e}^{\mathcal{X}_p(\tau)}$, then we obtain that $\mathcal{X}_P^{\Delta}(\tau) = (\ln u_p(\tau))^{\Delta}$. Thus system (1.1) becomes

$$(\ln u_p(\tau))^{\Delta} = r_p(\tau) - c_{p,1}(\tau)[u_p(\tau)]^{\theta_p} - \sum_{i=2}^{n+1} c_{p,i}(\tau) \int_{-\infty}^{0} k_{p,i-1}(s) u_p(\tau+s)\Delta s - \frac{\phi_p(\tau)}{u_p(\tau)}, \tau \in \mathbb{T}. \quad (4.1)$$

System (4.1) has at least one ω-periodic positive solution $(\widetilde{u}_1(\tau), \cdots, \widetilde{u}_p(\tau), \cdots, \widetilde{u}_m(\tau))^T \in \widetilde{\Omega}$ where

$$\widetilde{\Omega} = \Big\{(\widetilde{u}_1(\tau), \cdots, \widetilde{u}_p(\tau), \cdots, \widetilde{u}_m(\tau))^T : \mathrm{e}^{\alpha_{p_1} - \omega \overline{r_p}} < u_p(\tau) < \mathrm{e}^{\alpha_{p_2}},\ p = 1, 2, \cdots, m\}.$$

Let β and γ be some positive constants which satisfy $0 < \beta < \min\{\mathrm{e}^{\alpha_{p_1} - \omega \overline{r_p}}\}$, and $\gamma \geq \max\{1, \theta_1, \cdots,$
$\theta_p, \cdots, \theta_m\}$. Taking variable substitution $u_p(\tau) = (\beta X_p(\tau))^{\frac{1}{\gamma}}$, then we have

$$(\ln u_p(\tau))^{\Delta} = \left[\frac{1}{\gamma} \ln(\beta X_p(\tau))\right]^{\Delta} = \frac{1}{\gamma}\left[\ln \beta + \ln X_p(\tau)\right]^{\Delta} = \frac{1}{\gamma}(\ln X_p(\tau))^{\Delta}.$$

Consequently, system (4.1) becomes

$$\begin{aligned}(\ln X_p(\tau))^{\Delta} =& \gamma[r_p(\tau) - \beta^{\frac{\theta_p}{\gamma}} c_{p,1}(\tau) X^{\frac{\theta_p}{\gamma}}(\tau) - \sum_{i=2}^{n+1} \beta^{\frac{1}{\gamma}} c_{p,i}(\tau) \int_{-\infty}^{0} k_{p,i-1}(s) X_p^{\frac{1}{\gamma}}(\tau+s)\Delta s \\ & - \beta^{-\frac{1}{\gamma}} \phi_p(\tau) X_P^{-\frac{1}{\gamma}}(\tau)].\end{aligned} \quad (4.2)$$

Clearly, system (4.2) has one ω-periodic positive solution $(\widetilde{X}_1(\tau), \cdots, \widetilde{X}_p(\tau), \cdots, \widetilde{X}_m(\tau))^{\mathrm{T}} = (\frac{1}{\beta}\widetilde{u}_1^{\gamma}(\tau),$
$\cdots, \frac{1}{\beta}\widetilde{u}_p^{\gamma}(\tau), \cdots, \frac{1}{\beta}\widetilde{u}_m^{\gamma}(\tau))^{\mathrm{T}}$ within $\widetilde{\Omega}'$, here

$$\widetilde{\Omega}' = \{(X_1(\tau), \cdots, X_p(\tau), \cdots, X_m(\tau))^{\mathrm{T}} : \frac{1}{\beta}\mathrm{e}^{\gamma(\alpha_{p_1} - \omega \overline{r_p})} < X_p(\tau) < \frac{1}{\beta}\mathrm{e}^{\gamma \alpha_{p_2}}\}.$$

From Theorem 3.5 and $\widetilde{\Omega}'$, we have

$$1 < \frac{1}{\beta}\mathrm{e}^{\gamma(\alpha_{p_1} - \omega \overline{r_p})} < \widetilde{X}_p(\tau) < \frac{1}{\beta}\mathrm{e}^{\gamma \alpha_{p_2}}, p = 1, 2, \cdots, m. \quad (4.3)$$

Theorem 4.2. *If Assuption 1-Assuption 3 hold, in addition if*

$$-\beta^{\frac{\theta_p}{\gamma}} \underline{c_{p,1}} + \beta^{-\frac{1}{\gamma}} \overline{\phi_p} + \beta^{\frac{1}{\gamma}} \sum_{i=2}^{n+1} \overline{c_{p,i}} \int_{-\infty}^{0} k_{p,i-1}(s)\Delta s < 0. \quad (4.4)$$

then system (1.1) has one ω-periodic solution $(\widetilde{\mathcal{X}}_1(\tau), \cdots, \widetilde{\mathcal{X}}_p(\tau), \cdots, \widetilde{\mathcal{X}}_m(\tau))^T \in \Omega$ which is globally asymptotically stable and unique.

Proof. Since the global asymptotical stability of ω-periodic solution of (1.1) and (4.2) is equivalent, it suffices to prove that the ω-periodic solution $(\widetilde{X}_1(\tau), \cdots, \widetilde{X}_p(\tau), \cdots, \widetilde{X}_m(\tau))^{\mathrm{T}} \in \widetilde{\Omega}'$ of (4.2) is globally asymptotically stable. In fact, we know from Theorem 3.5 that $(\widetilde{X}_1(\tau), \cdots, \widetilde{X}_p(\tau), \cdots, \widetilde{X}_m(\tau))^{\mathrm{T}} \in$

$\widetilde{\Omega}'$ is an ω-periodic positive solution of (4.2). For any positive solution $(X_1(\tau), \cdots, X_p(\tau), \cdots, X_m(\tau))^{\mathrm{T}}$ of (4.2), we build a Lyapunov functional $V(\tau) = V_1(\tau) + V_2(\tau)$, where

$$V_1(\tau) = \sum_{p=1}^{m} |\ln X_p(\tau) - \ln \widetilde{X}_p(\tau)|, \tag{4.5}$$

$$V_2(\tau) = \sum_{p=1}^{m} \gamma\beta^{\frac{1}{\gamma}} \sum_{i=2}^{n+1} \overline{c_{p,i}} \int_{-\infty}^{0} k_{p,i-1}(s) \left[\int_{\tau+s}^{\tau} \left| X_P^{\frac{1}{\gamma}}(\zeta) - \widetilde{X}_p^{\frac{1}{\gamma}}(\zeta) \right| \Delta\zeta \right] \Delta s. \tag{4.6}$$

Obviously, $V(0) < +\infty$ and $V(\tau) \geq V_1(\tau)$. By (4.3), a direct Δ-derivation along (4.2) gives

$$\begin{aligned} & D^{+}(|\ln X_p(\tau) - \ln \widetilde{X}_p(\tau)|)^{\Delta} \\ \leq & -\gamma\beta^{\frac{\theta_p}{\gamma}} \underline{c_{p,1}} |X_p(\tau) - \widetilde{X}_p(\tau)| + \gamma\beta^{\frac{1}{\gamma}} \sum_{i=2}^{n+1} \overline{c_{p,i}} \int_{-\infty}^{0} k_{p,i-1}(s) |X_p^{\frac{1}{\gamma}}(\tau+s) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau+s)| \Delta s \\ & + \gamma\beta^{-\frac{1}{\gamma}} \overline{\phi_p} |X_p^{\frac{1}{\gamma}}(\tau) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau)|. \end{aligned} \tag{4.7}$$

Then, we can get the following results,

$$\begin{aligned} & D^{+}\left(\int_{-\infty}^{0} k_{p,i-1}(s) \left[\int_{\tau+s}^{\tau} \left| X_p^{\frac{1}{\gamma}}(\zeta) - \widetilde{X}_p^{\frac{1}{\gamma}}(\zeta) \right| \Delta\zeta \right] \Delta s \right)^{\Delta} \\ = & \int_{-\infty}^{0} k_{p,i-1}(s) \Delta s \cdot \left| X_p^{\frac{1}{\gamma}}(\tau) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau) \right| - \int_{-\infty}^{0} k_{p,i-1}(s) \left| X_p^{\frac{1}{\gamma}}(\tau+s) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau+s) \right| \Delta s, \end{aligned} \tag{4.8}$$

Notice that, for constants $a, b > 1$ and $s \geq 1$, $q(s) = |a^s - b^s|$ is monotonically increasing when $a \neq b$. Indeed, since $a \neq b$, without loss of generality, assume $a > b > 1$ (the proof is similar when $b > a > 1$). The monotonicity of $q(s) = a^s - b^s$ is the same as that of $q(s) = |a^s - b^s|$ when $a > b > 1$. Differentiate $q(s) = a^s - b^s$, we get $q'(s) = a^s lna - b^s lnb$. Because $y = lnx$ is monotonically increasing on $(0, +\infty)$, so $lna > lnb > 0$. Also, the function $y = x^m (m \geq 1)$ is monotonically increasing on $(0, +\infty)$, so for any s, $a^s > b^s > 0$. Then, $a^s lna - b^s lnb > 0$, that is, $q'(s) > 0$. Therefore, $q(s) = |a^s - b^s|$ is monotonically increasing when $a \neq b$. And

$0 < \frac{\theta_1}{\theta}, \frac{\theta_2}{\theta}, \frac{1}{\theta} \leq 1$, it follows from (4.3), (4.7) and (4.8) that

$$
\begin{aligned}
D^+V^{\Delta}(\tau) \leq & \sum_{p=1}^{m} \Big\{ -\gamma\beta^{\frac{\theta_p}{\gamma}} \underline{c_{p,1}} |X_p(\tau) - \widetilde{X}_p(\tau)| \\
& + \gamma\beta^{\frac{1}{\gamma}} \sum_{i=2}^{n+1} \overline{c_{p,i}} \int_{-\infty}^{0} k_{p,i-1}(s) |X_p^{\frac{1}{\gamma}}(\tau+s) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau+s)| \Delta s \\
& + \gamma\beta^{-\frac{1}{\gamma}} \overline{\phi_p} |X_p^{\frac{1}{\gamma}}(\tau) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau)| \\
& + \gamma\beta^{\frac{1}{\gamma}} \sum_{i=2}^{n+1} \overline{c_{p,i}} \int_{-\infty}^{0} k_{p,i-1}(s) \Delta s \cdot \left| X_p^{\frac{1}{\gamma}}(\tau) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau) \right| \\
& - \gamma\beta^{\frac{1}{\gamma}} \sum_{i=2}^{n+1} \overline{c_{p,i}} \int_{-\infty}^{0} k_{p,i-1}(s) \left| X_p^{\frac{1}{\gamma}}(\tau+s) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau+s) \right| \Delta s \Big\} \\
= & \sum_{p=1}^{m} \Big\{ -\gamma\beta^{\frac{\theta_p}{\gamma}} \underline{c_{p,1}} |X_p(\tau) - \widetilde{X}_p(\tau)| + \gamma\beta^{-\frac{1}{\gamma}} \overline{\phi_p} |X_p^{\frac{1}{\gamma}}(\tau) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau)| \\
& + \gamma\beta^{\frac{1}{\gamma}} \sum_{i=2}^{n+1} \overline{c_{p,i}} \int_{-\infty}^{0} k_{p,i-1}(s) \Delta s \cdot \left| X_p^{\frac{1}{\gamma}}(\tau) - \widetilde{X}_p^{\frac{1}{\gamma}}(\tau) \right| \Big\} \\
\leq & \sum_{p=1}^{m} \Big\{ -\gamma\beta^{\frac{\theta_p}{\gamma}} \underline{c_{p,1}} |X_p(\tau) - \widetilde{X}_p(\tau)| + \gamma\beta^{-\frac{1}{\gamma}} \overline{\phi_p} |X_p(\tau) - \widetilde{X}_p(\tau)| \\
& + \gamma\beta^{\frac{1}{\gamma}} \sum_{i=2}^{n+1} \overline{c_{p,i}} \int_{-\infty}^{0} k_{p,i-1}(s) \Delta s \cdot \left| X_p(\tau) - \widetilde{X}_p(\tau) \right| \Big\} \\
= & \sum_{p=1}^{m} \gamma \Big\{ \left[-\beta^{\frac{\theta_p}{\gamma}} \underline{c_{p,1}} + \beta^{-\frac{1}{\gamma}} \overline{\phi_p} + \beta^{\frac{1}{\gamma}} \sum_{i=2}^{n+1} \overline{c_{p,i}} \int_{-\infty}^{0} k_{p,i-1}(s) \Delta s \right] \times |X_p(\tau) - \widetilde{X}_p(\tau)| \Big\} < 0.
\end{aligned}
$$

Thus, from (4.5) and (4.6), we know that for all $\tau \geq 0, V(\tau)$ is positive definite and $D^+V^{\Delta}(\tau) < 0$. Therefore, according to Lyapunov stability theory, we conclude that the ω-periodic solution of (4.2) is globally asymptotically stable. Consequently, we know that the ω-periodic solution of (1.1) is globally asymptotically stable, then $(\widetilde{\mathcal{X}}_1(\tau), \cdots, \widetilde{\mathcal{X}}_p(\tau), \cdots, \widetilde{\mathcal{X}}_m(\tau))^{\mathrm{T}}$ is attractive, that is, for any solution $(\mathcal{X}_1(\tau), \cdots, \mathcal{X}_p(\tau), \cdots, \mathcal{X}_m(\tau))^{\mathrm{T}}$ of (1.1), we have $\lim\limits_{\tau \to +\infty} [\mathcal{X}_p(\tau) - \widetilde{\mathcal{X}}_p(\tau)] = 0$. If the system (1.1) has another ω-periodic solution $(\mathcal{X}_1^*(\tau), \cdots, \mathcal{X}_p^*(\tau), \cdots, \mathcal{X}_m^*(\tau))^{\mathrm{T}} \in \Omega$ such that $(\mathcal{X}_1^*(\tau), \cdots, \mathcal{X}_p^*(\tau), \cdots, \mathcal{X}_m^*(\tau))^{\mathrm{T}} \neq (\widetilde{\mathcal{X}}_1, (\tau), \cdots, \widetilde{\mathcal{X}}_p, (\tau), \cdots, \widetilde{\mathcal{X}}_m(\tau))^{\mathrm{T}}$. Without loss of generality, assuming that $\mathcal{X}_p^*(\tau) \neq \widetilde{\mathcal{X}}_p(\tau)$, we obtain $0 < |\widetilde{\mathcal{X}}_p(\tau) - \mathcal{X}_p^*(\tau)| \leq |\widetilde{\mathcal{X}}_P(\tau) - \mathcal{X}_P(\tau)| + |\mathcal{X}_p(\tau) - \mathcal{X}_p^*(\tau)| \to 0$, as $\tau \to +\infty$, which is an obvious fallacy. Thus, we prove that the ω-periodic solution of (1.1) is unique. The proof is completed. □

Next, we present the following example and verify the validity of the theoretical result through MATLAB calculations.

Example 4.3. We consider a nonlinear three-species GA predation ecosystem with infinite distributed lags on the time scale $\mathbb{T} = \mathbb{R}$:

$$\begin{cases} \frac{\mathrm{d}\mathcal{X}_1(\tau)}{\mathrm{d}\tau} = \mathcal{X}_1(\tau)\big[r_1(\tau) - c_{11}(\tau)\mathcal{X}_1^{\theta_1}(\tau) - c_{12}(\tau)\int_{-\infty}^{0} k_{11}(s)\mathcal{X}_1(\tau+s)\mathrm{d}s \\ \qquad - c_{13}(\tau)\int_{-\infty}^{0} k_{12}(s)\mathcal{X}_1(\tau+s)\mathrm{d}s - c_{14}(\tau)\int_{-\infty}^{0} k_{13}(s)\mathcal{X}_1(\tau+s)\mathrm{d}s\big] - \phi_1(\tau), \\ \frac{\mathrm{d}\mathcal{X}_2(\tau)}{\mathrm{d}\tau} = \mathcal{X}_2(\tau)\big[r_2(\tau) - c_{21}(\tau)\mathcal{X}_2^{\theta_2}(\tau) - c_{22}(\tau)\int_{-\infty}^{0} k_{21}(s)\mathcal{X}_2(\tau+s)\mathrm{d}s \\ \qquad - c_{23}(\tau)\int_{-\infty}^{0} k_{22}(s)\mathcal{X}_2(\tau+s)\mathrm{d}s - c_{24}(\tau)\int_{-\infty}^{0} k_{23}(s)\mathcal{X}_2(\tau+s)\mathrm{d}s\big] - \phi_2(\tau), \\ \frac{\mathrm{d}\mathcal{X}_3(\tau)}{\mathrm{d}\tau} = \mathcal{X}_3(\tau)\big[r_3(\tau) - c_{31}(\tau)\mathcal{X}_3^{\theta_3}(\tau) - c_{32}(\tau)\int_{-\infty}^{0} k_{31}(s)\mathcal{X}_3(\tau+s)\mathrm{d}s \\ \qquad - c_{33}(\tau)\int_{-\infty}^{0} k_{32}(s)\mathcal{X}_3(\tau+s)\mathrm{d}s - c_{34}(\tau)\int_{-\infty}^{0} k_{33}(s)\mathcal{X}_3(\tau+s)\mathrm{d}s\big] - \phi_3(\tau), \end{cases} \tag{4.9}$$

where

$$\begin{aligned} r_1(\tau) &= 10 + 2\cos(\tau), & r_2(\tau) &= 8 + \sin(\tau), & r_3(\tau) &= 6 + \cos(\tau), \\ c_{11}(\tau) &= 7 + \sin(\tau), & c_{21}(\tau) &= 5 + \cos(\tau), & c_{31}(\tau) &= 3 + \sin(\tau), \\ c_{12}(\tau) &= \frac{6+\sin(\tau)}{20}, & c_{13}(\tau) &= \frac{6+\sin(\tau)}{30}, & c_{14}(\tau) &= \frac{6+\sin(\tau)}{60}, \\ c_{22}(\tau) &= \frac{3+\cos(\tau)}{30}, & c_{23}(\tau) &= \frac{3+\cos(\tau)}{40}, & c_{24}(\tau) &= \frac{3+\cos(\tau)}{24}, \\ c_{32}(\tau) &= \frac{5+\cos(\tau)}{40}, & c_{33}(\tau) &= \frac{5+\cos(\tau)}{50}, & c_{34}(\tau) &= \frac{6+\cos(\tau)}{60}, \\ k_{11}(s) &= \frac{1}{2(1+s^2)}, & k_{12}(s) &= \frac{1}{3(1+s^2)}, & k_{13}(s) &= \frac{1}{6(1+s^2)}, \\ k_{21}(s) &= \frac{1}{3}\mathrm{e}^{2s}, & k_{22}(s) &= \frac{1}{4}\mathrm{e}^{2s}, & k_{23}(s) &= \frac{5}{12}\mathrm{e}^{2s}, \\ k_{31}(s) &= \frac{1}{3}\mathrm{e}^{s}, & k_{32}(s) &= \frac{1}{4}\mathrm{e}^{s}, & k_{33}(s) &= \frac{5}{12}\mathrm{e}^{s}, \\ \phi_1(\tau) &= \frac{3+\cos(\tau)}{7}, & \phi_2(\tau) &= \frac{4+\sin(\tau)}{7}, & \phi_3(\tau) &= \frac{5+\cos(\tau)}{7}, \\ \theta_1 &= \sqrt{3}, & \theta_2 &= \frac{1}{\sqrt{2}} & \theta_3 &= \sqrt{2}. \end{aligned}$$

Obviously, $r_p(\tau)$, $c_{ab}(\tau)(a = 1,2,3, b = 1,2,3,4)$ and $\phi_p(\tau)$ are all positive periodic functions with periodic $\omega = 2\pi$, $k_{ef}(s)(e = 1,2,3, f = 1,2,3) > 0$ and

$$\begin{aligned} &\int_{-\infty}^{0} k_{11}(s)\mathrm{d}s = \frac{\pi}{4} < +\infty, \quad \int_{-\infty}^{0} k_{12}(s)\mathrm{d}s = \frac{\pi}{6} < +\infty, \quad \int_{-\infty}^{0} k_{13}(s)\mathrm{d}s = \frac{\pi}{12} < +\infty, \\ &\int_{-\infty}^{0} k_{21}(s)\mathrm{d}s = \frac{1}{6} < +\infty, \quad \int_{-\infty}^{0} k_{22}(s)\mathrm{d}s = \frac{1}{8} < +\infty, \quad \int_{-\infty}^{0} k_{23}(s)\mathrm{d}s = \frac{5}{24} < +\infty, \\ &\int_{-\infty}^{0} k_{31}(s)\mathrm{d}s = \frac{1}{3} < +\infty, \quad \int_{-\infty}^{0} k_{32}(s)\mathrm{d}s = \frac{1}{4} < +\infty, \quad \int_{-\infty}^{0} k_{33}(s)\mathrm{d}s = \frac{5}{12} < +\infty, \end{aligned}$$

so the Assumption 1 and 2 hold. A direct computation gives

$$\begin{aligned}
&\overline{r_1}=12, \quad \underline{r_1}=8, \quad \overline{r_2}=9, \quad \underline{r_2}=7, \quad \overline{r_3}=7, \quad \underline{r_3}=5,\\
&\overline{c_{11}}=8, \quad \underline{c_{11}}=6, \quad \overline{c_{12}}=\frac{7}{20}, \quad \underline{c_{12}}=\frac{1}{4}, \quad \overline{c_{13}}=\frac{7}{30}, \quad \underline{c_{13}}=\frac{1}{6}, \quad \overline{c_{14}}=\frac{7}{60}, \quad \underline{c_{14}}=\frac{1}{12},\\
&\overline{c_{21}}=6, \quad \underline{c_{21}}=4, \quad \overline{c_{22}}=\frac{2}{15}, \quad \underline{c_{22}}=\frac{1}{15}, \quad \overline{c_{23}}=\frac{1}{10}, \quad \underline{c_{23}}=\frac{1}{20}, \quad \overline{c_{24}}=\frac{1}{6}, \quad \underline{c_{24}}=\frac{1}{12},\\
&\overline{c_{31}}=4, \quad \underline{c_{31}}=2, \quad \overline{c_{32}}=\frac{3}{20}, \quad \underline{c_{32}}=\frac{1}{10}, \quad \overline{c_{33}}=\frac{3}{25}, \quad \underline{c_{33}}=\frac{2}{25}, \quad \overline{c_{34}}=\frac{7}{60}, \quad \underline{c_{34}}=\frac{1}{12},\\
&\overline{\phi_1}=\frac{4}{7}, \quad \underline{\phi_1}=\frac{2}{7}, \quad \overline{\phi_2}=\frac{5}{7}, \quad \underline{\phi_2}=\frac{3}{7}, \quad \overline{\phi_3}=\frac{6}{7}, \quad \underline{\phi_3}=\frac{4}{7}.
\end{aligned}$$

In order to solve the following algebraic equation

$$f_1(Z)=(\underline{c_{11}}\mathrm{e}^{-\omega\theta_1\overline{r_1}})\mathrm{e}^{(1+\theta_1)Z}-\overline{r_1}\mathrm{e}^{Z}+\underline{\phi_1}=0,$$

we find that the solutions are $\alpha_{11}\approx-3.7377$ and $\alpha_{12}\approx 75.7984$. Similarly, for

$$f_2(Z)=(\underline{c_{21}}\mathrm{e}^{-\omega\theta_2\overline{r_2}})\mathrm{e}^{(1+\theta_2)Z}-\overline{r_2}\mathrm{e}^{Z}+\underline{\phi_2}=0,$$

we get $\alpha_{21}\approx-3.0445$ and $\alpha_{22}\approx 57.6995$.Similarly, for

$$f_3(Z)=(\underline{c_{31}}\mathrm{e}^{-\omega\theta_3\overline{r_3}})\mathrm{e}^{(1+\theta_3)Z}-\overline{r_3}\mathrm{e}^{Z}+\underline{\phi_3}=0,$$

we get $\alpha_{31}\approx-2.5055$ and $\alpha_{32}\approx 44.8681$. Thus we have gotten the following open bounded subsets $\Omega:\Omega=\left\{(\mathcal{X}_1(\tau),\mathcal{X}_2(\tau),\mathcal{X}_3(\tau))^T\right\}$, where $4.2826\times10^{-35}<\mathcal{X}_1(\tau)<8.2952\times10^{32}$,$1.3153\times10^{-26}<\mathcal{X}_2(\tau)<1.1444\times10^{25}$,$6.4658\times10^{-21}<\mathcal{X}_3(\tau)<3.0617\times10^{19}$ Then, we verify the Assumption 3 holds as follows:

$$\theta_1(\underline{c_{11}}\mathrm{e}^{-\omega\theta_1\overline{r_1}})^{-\frac{1}{\theta_1}}\left(\frac{\overline{r_1}}{1+\theta_1}\right)^{\frac{1+\theta_1}{\theta_1}}\approx 3.5325\times10^{33}>\underline{\phi_1}=\frac{2}{7},$$

$$\theta_2(\underline{c_{21}}\mathrm{e}^{-\omega\theta_2\overline{r_2}})^{-\frac{1}{\theta_2}}\left(\frac{\overline{r_2}}{1+\theta_2}\right)^{\frac{1+\theta_2}{\theta_2}}\approx 1.9945\times10^{25}>\underline{\phi_2}=\frac{3}{7},$$

$$\theta_3(\underline{c_{31}}\mathrm{e}^{-\omega\theta_3\overline{r_3}})^{-\frac{1}{\theta_3}}\left(\frac{\overline{r_3}}{1+\theta_3}\right)^{\frac{1+\theta_3}{\theta_3}}\approx 6.7322\times10^{19}>\underline{\phi_3}=\frac{4}{7}.$$

Now, we have verified the Assumption 1-Assumption 3. It follows from Theorem 3.5 that system (4.9) has at least one 2π-periodic positive solution $(\widetilde{\mathcal{X}}_1(\tau),\widetilde{\mathcal{X}}_2(\tau),\widetilde{\mathcal{X}}_3(\tau))^{\mathrm{T}}\in\Omega$. Next, we prove that the periodic positive solution is globally asymptotically stable and unique. Indeed, by taking

$\beta \approx 4.2 \times 10^{-35}, \gamma = 1000$, we get

$$
\begin{aligned}
&-\beta^{\frac{\theta_1}{\gamma}}\underline{c_{11}} + \beta^{-\frac{1}{\gamma}}\overline{\phi_1} + \beta^{\frac{1}{\gamma}}\overline{c_{12}}\int_{-\infty}^{0} k_{11}(s)\Delta s + \beta^{\frac{1}{\gamma}}\overline{c_{13}}\int_{-\infty}^{0} k_{12}(s)\Delta s \\
&+\beta^{\frac{1}{\gamma}}\overline{c_{14}}\int_{-\infty}^{0} k_{13}(s)\Delta s \approx -4.2148 < 0, \\
&-\beta^{\frac{\theta_2}{\gamma}}\underline{c_{21}} + \beta^{-\frac{1}{\gamma}}\overline{\phi_2} + \beta^{\frac{1}{\gamma}}\overline{c_{22}}\int_{-\infty}^{0} k_{21}(s)\Delta s + \beta^{\frac{1}{\gamma}}\overline{c_{23}}\int_{-\infty}^{0} k_{22}(s)\Delta s \\
&+\beta^{\frac{1}{\gamma}}\overline{c_{24}}\int_{-\infty}^{0} k_{23}(s)\Delta s \approx -2.7368 < 0, \\
&-\beta^{\frac{\theta_3}{\gamma}}\underline{c_{31}} + \beta^{-\frac{1}{\gamma}}\overline{\phi_3} + \beta^{\frac{1}{\gamma}}\overline{c_{32}}\int_{-\infty}^{0} k_{31}(s)\Delta s + \beta^{\frac{1}{\gamma}}\overline{c_{33}}\int_{-\infty}^{0} k_{32}(s)\Delta s \\
&+\beta^{\frac{1}{\gamma}}\overline{c_{34}}\int_{-\infty}^{0} k_{33}(s)\Delta s \approx -0.8059 < 0.
\end{aligned}
$$

Therefore, (4.4) holds. From Theorem 4.2, we can conclude that the periodic solution $(\widetilde{\mathcal{X}}_1(\tau), \widetilde{\mathcal{X}}_2(\tau), \widetilde{\mathcal{X}}_3(\tau))^{\mathrm{T}}$ is globally asymptotically stable and unique. The simulation results of the dynamic behavior of the solutions to system (4.9) are shown in the figure.

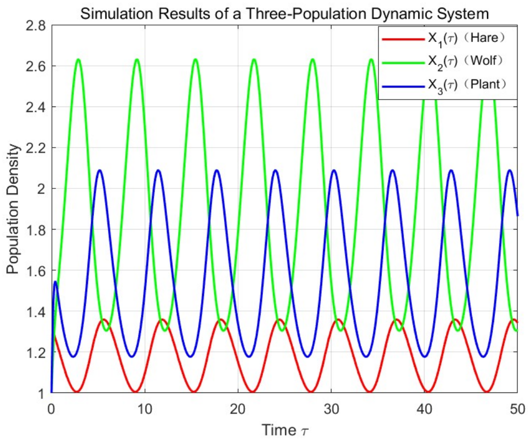

Figure 1

This figure illustrates the dynamic changes in the densities of multiple populations over time: all populations exhibit synchronous and stable periodic oscillation characteristics.

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Not applicable

Authors's Contributions

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