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Letters on Applied and Pure Mathematics

On Conformable differential equation with variable coefficients

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Abstract

In this paper, we study a differential equation involving a conformable derivative with variable coefficients. Using Banach's fixed-point theorem, we prove the global existence and uniqueness of a mild solution. We also establish the continuous dependence of the solution on the parameters. In addition, we show the convergence of the mild solution as the fractional order approaches 1^-.

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1 Introduction

In this paper, we study the following differential equation:

$$D_t^\alpha(y'(t)) + \lambda(t)y' = \mu G(D_t^\alpha y(t)), \quad y(0) = y_0. \tag{1.1}$$

The function $\lambda \in C([0,T])$ and the initial datum $y_0 \in \mathbb{R}$. The function G is defined later. The parameter μ is a real number. The symbol $\frac{^C\partial^\alpha}{\partial t^\alpha} f$ is defined by

$$\frac{^C\partial^\alpha}{\partial t^\alpha} f(t) := \lim_{\delta\to 0} \frac{f(t+\delta t^{1-\alpha}) - f(t)}{\delta}, \quad 0 < \alpha < 1, \tag{1.2}$$

which is understood as the conformable derivative of f. It is easy to see that the conformable derivative like (1.2) is exactly an extension of the classical derivative and was first established in [14]. Some models in practice need the appearance of conformable derivatives as clarified in [9, 22, 27]. In addition, the conformable derivative is well-behaved and obeys the Leibniz rule and chain rule.

As we know, fractional differential equations (FDE) have several fields of application, for example, in chemistry, theoretical biology and ecology, economics, and engineering. Conformable fractional differential equations have many applications in some different fields, for example, harmonic oscillations, damped oscillations (see [8]), electrical circuits (see [16]), some chaotic systems in dynamics (see [10]), motion of the projectile (see [3]). There are some results on ODEs and PDEs with fractional derivatives, for example, [1, 6, 11–13, 15, 18]. Conformable derivatives related to partial derivatives have been studied in many recent works, for example, [17, 21, 23]. In the following, we list some previous papers which mention our current paper.

In [7] M. Bouaouid, K. Hilal and S. Melliani showed the existence of mild solutions for a class of conformable fractional differential equations with nonlocal conditions. They applied semigroup theory combined with the Schaefer fixed point theorem. W. Zhong, L. Wang [26] considered the conformable fractional differential equation with integral boundary conditions. By applying the fixed point theorem in a cone, some criteria for the existence of at least one positive solution to the problem are established. B. Ahmad, M. Alghanmi, A. Alsaedi and R.V. Agarwal [2] studied a system of nonlinear conformable fractional differential equations with impulsive terms. By using the new tools of functional analysis, they showed the existence of the solution to their problem. Recently, Zhai and Liu [24] constructed a new conformable integro-differential equation with integral boundary conditions. By applying a recent fixed point theorem for monotone operators in ordered Banach spaces, they obtained the existence and uniqueness of positive solutions for the boundary value problem. For some results on initial boundary value problems of the conformable differential equations, we introduce some papers [5, 19, 25]. These three works refer to the existence of solutions to boundary value problems for some specific conformable differential equations. Some methods were applied for proving the existence of boundary value problems, for example, tube solution and Schauder's fixed-point theorem [5], Krasnoselskii's fixed point theorem [4].

To the best of our knowledge, there are not any results mentioned to problem (1.1). Our main results are described as follows. Under the global Lipschitz assumption on G, we obtain the global existence of a mild solution. The second result concerns the continuity of the mild solution with the coefficient λ and μ. We also get the convergence of the mild solution when $\alpha \to 1^-$.

2 Main results

In order to construct the solution, we need to introduce the following space. Let us define $C^\theta[0,T]$ such that

$$\left\|v\right\|_{C^\theta([0,T])} = \sup_{0\le t\le T} e^{\frac{-\theta t^\alpha}{\alpha}} |v(t)|.$$

Theorem 2.1. *Let G be a function such that $G(0)=0$ and*

$$|G(a)-G(b)| \le K|a-b|, \qquad \forall a,b\in\mathbb{R},$$

for some constant $K>0$. Assume moreover that $\lambda \in C([0,T])$ and $\lambda(t)\ge\lambda_0>0$ for all $t\in[0,T]$. Then problem (1.1) has a unique mild solution $y\in C^1([0,T])$ with $y'\in C^\theta([0,T])$, where $\theta = 2T|\mu|K$. In addition, we get

$$|y(t)| \le 2Te^{\frac{2T\mu K T^\alpha}{\alpha}}|y_1| + |y_0|.$$

Proof. Since the fact that

$$D_t^\alpha y(t) = t^{1-\alpha} y'(t),$$

The main equation becomes to

$$t^{1-\alpha}y''(t) + \lambda y' = \mu G(t^{1-\alpha}y'(t)).$$

This implies that

$$y''(t) + \lambda t^{\alpha-1}y' = \mu t^{\alpha-1}G(t^{1-\alpha}y'(t)).$$

Let $z(t) = y'(t)$. Then we get the following equality

$$z'(t) + \lambda(t)t^{\alpha-1}z(t) = \mu t^{\alpha-1}G\Big(t^{1-\alpha}z(t)\Big), \quad z(0) = y'(0) = y_1.$$

Then one has

$$\begin{aligned} z(t) &= \exp\Big(-\int_0^t \lambda(s)s^{\alpha-1}ds\Big)y_1 \\ &\quad + \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)G\Big(s^{1-\alpha}z(s)\Big)ds. \end{aligned} \tag{2.1}$$

Our next goal is to show that problem (2.1) has a unique solution z. We set $\mathscr{Q} : C^\theta([0,T]) \to C^\theta([0,T])$ as follows

$$\begin{aligned} \mathscr{Q}z(t) &= \exp\Big(-\int_0^t \lambda(s)s^{\alpha-1}ds\Big)y_1 \\ &\quad + \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)G\Big(s^{1-\alpha}z(s)\Big)ds. \end{aligned}$$

Take the zero function $z_0 = 0$. Then since the fact that $G(0) = 0$, we get

$$\mathscr{Q}z_0(t) = \exp\Big(-\int_0^t \lambda(s)s^{\alpha-1}ds\Big)y_1.$$

Then we get that

$$e^{\frac{-\theta t^\alpha}{\alpha}}\Big|\mathscr{Q}z_0(t)\Big| \le |y_1|,$$

which allows us to deduce that $\mathscr{Q}z_0 \in C^\theta([0,T])$ and

$$\Big\|\mathscr{Q}z_0\Big\|_{C^\theta([0,T])} \le |y_1|.$$

Let us take any functions $z_1, z_2 \in C^\theta([0,T])$. Then we get

$$\begin{aligned} &\mathscr{Q}z_1(t) - \mathscr{Q}z_2(t) \\ &\quad = \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)\Big[G\Big(s^{1-\alpha}z_1(s)\Big) - G\Big(s^{1-\alpha}z_2(s)\Big)\Big]ds. \end{aligned}$$

Since the fact that $\lambda(r) \ge \lambda_0 > 0$, we know that

$$\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big) \le \exp\Big(-\lambda_0\frac{t^\alpha - s^\alpha}{\alpha}\Big). \tag{2.2}$$

Using global Lipschitz of G, we know that

$$\Big|G\Big(s^{1-\alpha}z_1(s)\Big) - G\Big(s^{1-\alpha}z_2(s)\Big)\Big| \le Ks^{1-\alpha}\Big|z_1(s) - z_2(s)\Big|.$$

This implies that

$$\begin{aligned}\Big|\mathscr{Q}z_1(t)-\mathscr{Q}z_2(t)\Big| &\le K\mu\int_0^t \exp\Big(-\lambda_0\frac{t^\alpha-s^\alpha}{\alpha}\Big)\Big|z_1(s)-z_2(s)\Big|ds\\ &= K\mu\int_0^t \exp\Big(-\lambda_0\frac{t^\alpha-s^\alpha}{\alpha}\Big)e^{\frac{\theta s^\alpha}{\alpha}}e^{\frac{-\theta s^\alpha}{\alpha}}\Big|z_1(s)-z_2(s)\Big|ds\\ &\le K\mu\Big(\int_0^t \exp\Big(-\lambda_0\frac{t^\alpha-s^\alpha}{\alpha}\Big)e^{\frac{\theta s^\alpha}{\alpha}}ds\Big)\Big\|z_1-z_2\Big\|_{C^\theta([0,T])}.\end{aligned}$$

Thus, we obtain that

$$e^{\frac{-\theta t^\alpha}{\alpha}}\Big|\mathscr{Q}z_1(t)-\mathscr{Q}z_2(t)\Big| \le K\mu\Big(\int_0^t \exp\Big(-(\lambda_0+\theta)\frac{t^\alpha-s^\alpha}{\alpha}\Big)ds\Big)\Big\|z_1-z_2\Big\|_{C^\theta([0,T])}. \tag{2.3}$$

Let us treat the integral term on the right above. Indeed, we find that

$$\begin{aligned}&\int_0^t \exp\Big(-(\lambda_0+\theta)\frac{t^\alpha-s^\alpha}{\alpha}\Big)ds\\ &\quad= \int_0^t s^{1-\alpha}s^{\alpha-1}\exp\Big(-(\lambda_0+\theta)\frac{t^\alpha-s^\alpha}{\alpha}\Big)ds\\ &\quad\le T^{1-\alpha}\int_0^t s^{\alpha-1}\exp\Big(-(\lambda_0+\theta)\frac{t^\alpha-s^\alpha}{\alpha}\Big)ds\\ &\quad= T^{1-\alpha}\exp\Big(-(\lambda_0+\theta)\frac{t^\alpha}{\alpha}\Big)\int_0^t s^{\alpha-1}\exp\Big(-(\lambda_0+\theta)\frac{s^\alpha}{\alpha}\Big)ds.\end{aligned}$$

It is obvious to see that

$$\int_0^t s^{\alpha-1}\exp\Big(-(\lambda_0+\theta)\frac{s^\alpha}{\alpha}\Big)ds = \frac{\exp\Big((\lambda_0+\theta)\frac{t^\alpha}{\alpha}\Big)-1}{\lambda_0+\theta}.$$

Hence, we find that

$$\int_0^t \exp\Big(-(\lambda_0+\theta)\frac{t^\alpha-s^\alpha}{\alpha}\Big)ds \le T^{1-\alpha}\frac{1-\exp\Big(-(\lambda_0+\theta)\frac{t^\alpha}{\alpha}\Big)}{\lambda_0+\theta}. \tag{2.4}$$

Combining (2.3) and (2.4), we find that

$$e^{\frac{-\theta t^\alpha}{\alpha}}\Big|\mathscr{Q}z_1(t)-\mathscr{Q}z_2(t)\Big| \le \frac{T\mu K}{\lambda_0+\theta}\Big\|z_1-z_2\Big\|_{C^\theta([0,T])}.$$

Hence, we get that

$$\Big\|\mathscr{Q}z_1-\mathscr{Q}z_2\Big\|_{C^\theta([0,T])} \le \frac{T\mu K}{\lambda_0+\theta}\Big\|z_1-z_2\Big\|_{C^\theta([0,T])}.$$

Let us choose $\theta = 2T|\mu|K$, we find that

$$\Big\|\mathscr{Q}z_1-\mathscr{Q}z_2\Big\|_{C^\theta([0,T])} \le \frac{1}{2}\Big\|z_1-z_2\Big\|_{C^\theta([0,T])}.$$

Since the above estimates, we can conclude that $\mathscr{Q}$ is a contraction on the space $C^\theta([0,T])$ for $\theta = 2T\mu K$. Thus, there exists a function $z \in C^\theta([0,T])$ which satisfies that

$$\begin{aligned}z(t) &= \exp\Big(-\int_0^t \lambda(s)s^{\alpha-1}ds\Big)y_1\\ &\quad+\mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)G\Big(s^{1-\alpha}z(s)\Big)ds.\end{aligned}$$

Since the appearance of z, we can choose y such that

$$y(t) = \int_0^t z(s)ds + y_0.$$

The above equality claims the existence of the mild solution y. In addition, we get

$$\begin{aligned}\Big\|z\Big\|_{C^\theta([0,T])} &\le \Big\|\mathscr{Q}z - \mathscr{Q}(z=0)\Big\|_{C^\theta([0,T])} + \Big\|\mathscr{Q}(z=0)\Big\|_{C^\theta([0,T])} \\ &\le \frac{1}{2}\Big\|z\Big\|_{C^\theta([0,T])} + |y_1|.\end{aligned}$$

This implies that

$$|z(t)| \le 2e^{\frac{2T\mu K t^\alpha}{\alpha}}|y_1|. \tag{2.5}$$

Thus, we find that

$$|y(t)| \le \int_0^t |z(s)|ds + |y_0| \le 2Te^{\frac{2T\mu K T^\alpha}{\alpha}}|y_1| + |y_0|.$$

□

Theorem 2.2. *Let y_λ and $y_{\widetilde{\lambda}}$ be two solutions of Problem (1.1) with the parameter λ and $\widetilde{\lambda}$. Then we get*

$$\Big\|y_\lambda - y_{\widetilde{\lambda}}\Big\|_{L^\infty(0,T)} \le Te^{K\mu T}\left(\frac{T^\alpha}{\alpha}e^{\lambda_0\frac{T^\alpha}{\alpha}} + \frac{\mu T^{\alpha+1}}{\alpha(\alpha+1)}e^{\frac{2T\mu K T^\alpha}{\alpha}+\lambda_0\frac{T^\alpha}{\alpha}}\right)\Big\|\lambda - \widetilde{\lambda}\Big\|_{L^\infty(0,T)}|y_1|. \tag{2.6}$$

Proof. Let us recall that z_λ be as follows

$$\begin{aligned}z_\lambda(t) &= \exp\Big(-\int_0^t \lambda(s)s^{\alpha-1}ds\Big)y_1 \\ &+ \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)G\Big(s^{1-\alpha}z_\lambda(s)\Big)ds.\end{aligned}$$

and

$$\begin{aligned}z_{\widetilde{\lambda}}(t) &= \exp\Big(-\int_0^t \widetilde{\lambda}(s)s^{\alpha-1}ds\Big)y_1 \\ &+ \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \widetilde{\lambda}(r)r^{\alpha-1}dr\Big)G\Big(s^{1-\alpha}z_{\widetilde{\lambda}}(s)\Big)ds.\end{aligned}$$

From the two above equalities, we find that

$$\begin{aligned}&z_\lambda(t) - z_{\widetilde{\lambda}}(t) = \exp\Big(-\int_0^t \lambda(s)s^{\alpha-1}ds\Big)y_1 - \exp\Big(-\int_0^t \widetilde{\lambda}(s)s^{\alpha-1}ds\Big)y_1 \\ &+ \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)\bigg(G\Big(s^{1-\alpha}z_\lambda(s)\Big) - G\Big(s^{1-\alpha}z_{\widetilde{\lambda}}(s)\Big)\bigg)ds \\ &+ \mu\int_0^t s^{\alpha-1}\bigg[\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big) - \exp\Big(-\int_s^t \widetilde{\lambda}(r)r^{\alpha-1}dr\Big)\bigg]G\Big(s^{1-\alpha}z_{\widetilde{\lambda}}(s)\Big)ds \qquad (2.7) \\ &= (A) + (B) + (C). \qquad (2.8)\end{aligned}$$

Let us consider the first term (A). Indeed, using the inequality $|e^{-a} - e^{-b}| \leq |a-b|$ for any $a, b \geq 0$, we find that

$$\begin{aligned}|(A)| &\leq \left| \int_0^t \Big(\lambda(s) - \widetilde{\lambda}(s)\Big) s^{\alpha-1} ds \right| |y_1| \\ &\leq |y_1| \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} \int_0^t s^{\alpha-1} ds = \frac{T^\alpha}{\alpha} \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} |y_1|. \end{aligned} \tag{2.9}$$

We continue to treat the second term (B). Since (2.2), we get that

$$\begin{aligned}|(B)| &\leq \mu \int_0^t s^{\alpha-1} \exp\Big(-\lambda_0 \frac{t^\alpha - s^\alpha}{\alpha}\Big) \left| G\Big(s^{1-\alpha} z_\lambda(s)\Big) - G\Big(s^{1-\alpha} z_{\widetilde{\lambda}}(s)\Big) \right| ds \\ &\leq K\mu \int_0^t \exp\Big(-\lambda_0 \frac{t^\alpha - s^\alpha}{\alpha}\Big) \left| z_\lambda(s) - z_{\widetilde{\lambda}}(s) \right| ds. \end{aligned} \tag{2.10}$$

Finally, we treat the term (C). In view of the inequality $|e^{-a} - e^{-b}| \leq |a-b|$ for any $a, b \geq 0$, we obtain

$$\begin{aligned}&\left| \exp\Big(-\int_s^t \lambda(r) r^{\alpha-1} dr \Big) - \exp\Big(-\int_s^t \widetilde{\lambda}(r) r^{\alpha-1} dr \Big) \right| \leq \left| \int_s^t \lambda(r) r^{\alpha-1} dr - \int_s^t \widetilde{\lambda}(r) r^{\alpha-1} dr \right| \\ &\qquad\qquad \leq \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} \int_s^t r^{\alpha-1} dr = \frac{t^\alpha - s^\alpha}{\alpha} \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)}. \end{aligned}$$

This implies that

$$|(C)| \leq \mu \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} \int_0^t s^{\alpha-1} \left(\frac{t^\alpha - s^\alpha}{\alpha} \right) \left| G\Big(s^{1-\alpha} z_{\widetilde{\lambda}}(s)\Big) \right| ds. \tag{2.11}$$

Using (2.5) and globally Lipschitz of G, we find that

$$\left| G\Big(s^{1-\alpha} z_{\widetilde{\lambda}}(s)\Big) \right| \leq K s^{1-\alpha} |z_{\widetilde{\lambda}}(s)| \leq K s^{1-\alpha} e^{\frac{2T\mu K T^\alpha}{\alpha}} |y_1|. \tag{2.12}$$

By collecting (2.11) and (2.12), we find that

$$\begin{aligned}|(C)| &\leq \mu e^{\frac{2T\mu K T^\alpha}{\alpha}} |y_1| \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} \int_0^t \left(\frac{t^\alpha - s^\alpha}{\alpha} \right) ds \\ &\leq \frac{\mu T^{\alpha+1}}{\alpha(\alpha+1)} e^{\frac{2T\mu K T^\alpha}{\alpha}} |y_1| \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)}. \end{aligned} \tag{2.13}$$

Combining (2.7), (2.9), (2.10) and (2.13), we deduce that

$$\begin{aligned}\left| z_\lambda(t) - z_{\widetilde{\lambda}}(t) \right| &\leq |(A)| + |(B)| + |(C)| \\ &\leq \frac{T^\alpha}{\alpha} \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} |y_1| + \frac{\mu T^{\alpha+1}}{\alpha(\alpha+1)} e^{\frac{2T\mu K T^\alpha}{\alpha}} |y_1| \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} \\ &\quad + K\mu \int_0^t \exp\Big(-\lambda_0 \frac{t^\alpha - s^\alpha}{\alpha}\Big) \left| z_\lambda(s) - z_{\widetilde{\lambda}}(s) \right| ds. \end{aligned}$$

Multiplying both sides to $e^{\lambda_0 \frac{t^\alpha}{\alpha}}$, we obtain that

$$\begin{aligned}e^{\lambda_0 \frac{t^\alpha}{\alpha}} \left| z_\lambda(t) - z_{\widetilde{\lambda}}(t) \right| &\leq \left(\frac{T^\alpha}{\alpha} e^{\lambda_0 \frac{T^\alpha}{\alpha}} + \frac{\mu T^{\alpha+1}}{\alpha(\alpha+1)} e^{\frac{2T\mu K T^\alpha}{\alpha} + \lambda_0 \frac{T^\alpha}{\alpha}} \right) \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} |y_1| \\ &\quad + K\mu \int_0^t e^{\lambda_0 \frac{s^\alpha}{\alpha}} \left| z_\lambda(s) - z_{\widetilde{\lambda}}(s) \right| ds. \end{aligned}$$

By applying Gronwall's inequality, we deduce that

$$e^{\lambda_0 \frac{t^\alpha}{\alpha}} \left| z_\lambda(t) - z_{\widetilde{\lambda}}(t) \right| \le e^{K\mu t} \left(\frac{T^\alpha}{\alpha} e^{\lambda_0 \frac{T^\alpha}{\alpha}} + \frac{\mu T^{\alpha+1}}{\alpha(\alpha+1)} e^{\frac{2T\mu K T^\alpha}{\alpha} + \lambda_0 \frac{T^\alpha}{\alpha}} \right) \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} |y_1|.$$

Hence, we have immediately that

$$\left| z_\lambda(t) - z_{\widetilde{\lambda}}(t) \right| \le e^{K\mu t} \left(\frac{T^\alpha}{\alpha} e^{\lambda_0 \frac{T^\alpha}{\alpha}} + \frac{\mu T^{\alpha+1}}{\alpha(\alpha+1)} e^{\frac{2T\mu K T^\alpha}{\alpha} + \lambda_0 \frac{T^\alpha}{\alpha}} \right) \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} |y_1|. \tag{2.14}$$

Moreover, we recall the formula

$$y_\lambda(t) = \int_0^t z_\lambda(s) ds + y_0,$$

and

$$y_{\widetilde{\lambda}}(t) = \int_0^t z_{\widetilde{\lambda}}(s) ds + y_0.$$

Then we follow from (2.14) that

$$\begin{aligned} \left| y_\lambda(t) - y_{\widetilde{\lambda}}(t) \right| &\le \int_0^t \left| z_\lambda(s) - z_{\widetilde{\lambda}}(s) \right| ds \\ &\le T e^{K\mu T} \left(\frac{T^\alpha}{\alpha} e^{\lambda_0 \frac{T^\alpha}{\alpha}} + \frac{\mu T^{\alpha+1}}{\alpha(\alpha+1)} e^{\frac{2T\mu K T^\alpha}{\alpha} + \lambda_0 \frac{T^\alpha}{\alpha}} \right) \left\| \lambda - \widetilde{\lambda} \right\|_{L^\infty(0,T)} |y_1| \end{aligned}$$

which shows that (2.6). The proof of Theorem is completed. □

In the following theorem, we provide a theorem which shows that the convergence of the mild solution to problem (1.1).

Theorem 2.3. *Let y_α be the mild solution of problem (1.1). Let y^* be the mild solution of the following problem*

$$y''(t) + \lambda(t) y' = \mu G(y'(t)), \quad y(0) = y_0. \tag{2.15}$$

Let us assume that $0 < \lambda_0 \le \lambda(t) \le M_0$. Then we get

$$\left| y_\alpha(t) - y^*(t) \right| \le \widetilde{C} \Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| + \frac{1-\alpha}{\alpha} \Big] |y_1| e^{K\mu T} T.$$

Here $0 < \mu \le \alpha_0$ and $\widetilde{C}$ is a positive constant which depends on $\mu, K, M_0, T, \alpha, \mu_0$.

Proof. Let us first consider the solution of problem (2.15). Let $w^*(t) = (y^*)'(t)$. Then we get

$$(w^*)'(t) + \lambda(t) w^*(t) = \mu G(w^*(t)), \quad w(0) = y_1.$$

Then we obtain

$$w^*(t) = \exp\Big(- \int_0^t \lambda(s) ds \Big) y_1 + \mu \int_0^t \exp\Big(- \int_s^t \lambda(r) dr \Big) G\Big(w^*(s) \Big) ds, \tag{2.16}$$

and

$$y^*(t) = \int_0^t w^*(s)ds + y_0. \tag{2.17}$$

By a similar proof of Theorem 2.2, we show that problem (2.16) has a unique mild solution w^*. Our next aim is to give the upper bound for the term

$$\Big|z_\alpha(t) - w^*(t)\Big|,$$

where

$$\begin{aligned} z_\alpha(t) &= \exp\Big(-\int_0^t \lambda(s)s^{\alpha-1}ds\Big)y_1 \\ &\quad + \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)G\Big(s^{1-\alpha}z_\alpha(s)\Big)ds. \end{aligned} \tag{2.18}$$

From (2.16) and (2.18), we obtain that

$$\begin{aligned} z_\alpha(t) - w^*(t) &= \Bigg[\exp\Big(-\int_0^t \lambda(s)s^{\alpha-1}ds\Big) - \exp\Big(-\int_0^t \lambda(s)ds\Big)\Bigg]y_1 \\ &\quad + \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)\Big[G\Big(s^{1-\alpha}z_\alpha(s)\Big) - G\Big(w^*(s)\Big)\Big]ds \\ &\quad + \mu\int_0^t \Bigg[s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big) - s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)dr\Big)\Bigg]G\Big(w^*(s)\Big)ds \\ &\quad + \mu\int_0^t \Big(s^{\alpha-1}-1\Big)\exp\Big(-\int_s^t \lambda(r)dr\Big)G\Big(w^*(s)\Big)ds \\ &= \mathscr{K}_1 + \mathscr{K}_2 + \mathscr{K}_3 + \mathscr{K}_4. \end{aligned}$$

Step 1. Estimate of $\mathscr{K}_1$.
Let us consider the term $\mathscr{K}_1$. In view of the inequality $|e^{-a} - e^{-b}| \le |a-b|$ for any $a, b \ge 0$, we obtain that

$$\Big|\mathscr{K}_1\Big| \le \Big|\int_0^t \lambda(s)s^{\alpha-1}ds - \int_0^t \lambda(s)ds\Big| \le M_0\int_0^t |s^{\alpha-1}-1|ds$$

where we have used the fact that $\lambda(s) \le M_0$ for $0 \le s \le T$. By the result [20], we obtain that

$$\int_0^t |s^{\alpha-1}-1|ds \le C(\alpha_0,\mu)t^{\alpha-\mu_0}\Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha}-1|\Big] + \frac{1-\alpha}{\alpha}, \tag{2.19}$$

where $0 < \mu \le \alpha_0$. Thus, we have immediately that

$$\Big|\mathscr{K}_1\Big| \le C_1\Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha}-1| + \frac{1-\alpha}{\alpha}\Big].$$

Here C_1 depends on $M_0, \mu, \alpha_0, \mu, T$.
Step 2. Estimate of $\mathscr{K}_2$.
It is obvious to see that

$$\begin{aligned} \mathscr{K}_2 &= \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)\Big[G\Big(s^{1-\alpha}z_\alpha(s)\Big) - G\Big(s^{1-\alpha}w^*(s)\Big)\Big]ds \\ &\quad + \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)\Big[G\Big(s^{1-\alpha}w^*(s)\Big) - G\Big(w^*(s)\Big)\Big]ds \\ &= \mathscr{K}_{2,1} + \mathscr{K}_{2,2}. \end{aligned}$$

Using globally Lipschitz of G and (2.2), we get

$$\begin{aligned}\mathscr{K}_{2,1} &\leq K\mu \int_0^t s^{\alpha-1} \exp\Big(-\lambda_0 \frac{t^\alpha - s^\alpha}{\alpha}\Big) s^{1-\alpha} \Big| z_\alpha(s) - w^*(s) \Big| ds \\ &\leq K\mu \int_0^t \Big| z_\alpha(s) - w^*(s) \Big| ds.\end{aligned}$$

In addition, in view of (2.2), we get

$$\begin{aligned}\mathscr{K}_{2,2} &\leq K\mu \int_0^t s^{\alpha-1} \exp\Big(-\lambda_0 \frac{t^\alpha - s^\alpha}{\alpha}\Big) |s^{1-\alpha} - 1| \Big| w^*(s) \Big| ds \\ &\leq KC\mu|y_1| \int_0^t |1 - s^{\alpha-1}|\ ds\end{aligned}$$

where we note that $\Big|w^*(s)\Big| \leq C|y_1|$. Thus, we use (2.19) to obtain that

$$\mathscr{K}_{2,2} \leq KC\mu|y_1| \Bigg[C(\alpha_0,\mu) T^{\alpha-\mu_0} \Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| \Big] + \frac{1-\alpha}{\alpha} \Bigg].$$

Since the two above observations, we derive that

$$\begin{aligned}\mathscr{K}_2 &\leq \mathscr{K}_{2,1} + \mathscr{K}_{2,2} \\ &\leq C_2|y_1| \Bigg[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| + \frac{1-\alpha}{\alpha} \Bigg] \\ &+ K\mu \int_0^t \Big| z_\alpha(s) - w^*(s) \Big| ds.\end{aligned}$$

Here C_2 depends on $K, C, \mu, \alpha_0, \mu, T$.

Step 3. Estimate of $\mathscr{K}_3$. In view of the inequality $|e^{-a} - e^{-b}| \leq |a - b|$ for any $a, b \geq 0$, we find that

$$\begin{aligned}\Big| \exp\Big(-\int_s^t \lambda(r) r^{\alpha-1} dr\Big) - \exp\Big(-\int_s^t \lambda(r) dr\Big) \Big| &\leq \int_s^t \lambda(r) |r^{\alpha-1} - 1| dr \\ &\leq M_0 \int_s^t |r^{\alpha-1} - 1| dr \leq M_0 \int_0^t |r^{\alpha-1} - 1| dr.\end{aligned}$$

Using (2.19), we derive that

$$\begin{aligned}&\Big| \exp\Big(-\int_s^t \lambda(r) r^{\alpha-1} dr\Big) - \exp\Big(-\int_s^t \lambda(r) dr\Big) \Big| \\ &\qquad \leq M_0 \Bigg[C(\alpha_0,\mu) T^{\alpha-\mu_0} \Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| \Big] + \frac{1-\alpha}{\alpha} \Bigg].\end{aligned}$$

This implies that

$$\begin{aligned}\mathscr{K}_3 &\leq C_3|y_1| \Bigg[C(\alpha_0,\mu) T^{\alpha-\mu_0} \Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| \Big] + \frac{1-\alpha}{\alpha} \Bigg] \int_0^t s^{\alpha-1} ds \\ &\leq C_3|y_1| \Bigg[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| + \frac{1-\alpha}{\alpha} \Bigg],\end{aligned}$$

where C_3 depends on $\mu, K, M_0, T, \alpha, \mu_0$.
Step 4. Estimate of $\mathscr{K}_4$.
By similar techniques as above, we find that

$$\begin{aligned}|\mathscr{K}_4| &\le \mu \int_0^t \left|s^{\alpha-1} - 1\right|\left|G\Big(w^*(s)\Big)\right| ds \\ &\le C_4\Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| + \frac{1-\alpha}{\alpha}\Big].\end{aligned}$$

Here C_4 depends on M_0, α_0, μ, T. By collecting the four steps as above, we deduce that

$$\begin{aligned}\left|z_\alpha(t) - w^*(t)\right| &\le |\mathscr{K}_1| + |\mathscr{K}_2| + |\mathscr{K}_3| + |\mathscr{K}_4| \\ &\le \widetilde{C}\Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| + \frac{1-\alpha}{\alpha}\Big]|y_1| + K\mu \int_0^t \left|z_\alpha(s) - w^*(s)\right| ds,\end{aligned}$$

where $\widetilde{C}$ depends on $\mu, K, M_0, T, \alpha, \mu_0$. By applying Gronwall's inequality, we claim that

$$\left|z_\alpha(t) - w^*(t)\right| \le \widetilde{C}\Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| + \frac{1-\alpha}{\alpha}\Big]|y_1| e^{K\mu t}. \tag{2.20}$$

Using (2.17) and noting that

$$y_\alpha(t) = \int_0^t z_\alpha(s) ds + y_0$$

we know that

$$y_\alpha(t) - y^*(t) = \int_0^t \left(z_\alpha(s) - w^*(s)\right) ds. \tag{2.21}$$

Combining (2.20) and (2.21), we obtain that

$$\left|y_\alpha(t) - y^*(t)\right| \le \widetilde{C}\Big[(1-\alpha)^\mu + (1-\alpha) + |T^{1-\alpha} - 1| + \frac{1-\alpha}{\alpha}\Big]|y_1| e^{K\mu T} t.$$

The proof is completed.

□

Theorem 2.4. *Let y_μ and $y_{\mu'}$ be two mild solutions to problem (1.1). Then we get*

$$\left|y_\mu(t) - y_{\mu'}(t)\right| \le 2T^2 e^{\frac{2T\mu K T^\alpha}{\alpha}} e^{K\mu T} |y_1||\mu - \mu'|.$$

Proof. Let y_μ and $y_{\mu'}$ be two mild solutions to problem (1.1). Then we get

$$y_\mu(t) = \int_0^t z_\mu(s) ds + y_0, \quad y_{\mu'}(t) = \int_0^t z_{\mu'}(s) ds + y_0. \tag{2.22}$$

Here two functions z_μ and $z_{\mu'}$ satisfy that

$$\begin{aligned}z_\mu(t) &= \exp\Big(-\int_0^t \lambda(s) s^{\alpha-1} ds\Big) y_1 \\ &\quad + \mu \int_0^t s^{\alpha-1} \exp\Big(-\int_s^t \lambda(r) r^{\alpha-1} dr\Big) G\Big(s^{1-\alpha} z_\mu(s)\Big) ds,\end{aligned} \tag{2.23}$$

and

$$z_{\mu'}(t) = \exp\Big(-\int_0^t \lambda(s)s^{\alpha-1}ds\Big)y_1 \\ + \mu'\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)G\Big(s^{1-\alpha}z_{\mu'}(s)\Big)ds. \tag{2.24}$$

From (2.23) and (2.24), we derive that

$$\begin{aligned} & z_\mu(t) - z_{\mu'}(t) \\ & = \mu\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)\Big[G\Big(s^{1-\alpha}z_\mu(s)\Big) - G\Big(s^{1-\alpha}z_{\mu'}(s)\Big)\Big]ds \\ & + \Big(\mu-\mu'\Big)\int_0^t s^{\alpha-1}\exp\Big(-\int_s^t \lambda(r)r^{\alpha-1}dr\Big)G\Big(s^{1-\alpha}z_{\mu'}(s)\Big)ds \\ & = \mathscr{P}_1 + \mathscr{P}_2. \end{aligned} \tag{2.25}$$

Let us treat the first term $\mathscr{P}_1$. Using global Lipschitz property of G, we find that

$$\Big|\mathscr{P}_1\Big| \le K\mu\int_0^t s^{\alpha-1}s^{1-\alpha}\Big|z_\mu(s) - z_{\mu'}(s)\Big|ds = K\mu\int_0^t \Big|z_\mu(s) - z_{\mu'}(s)\Big|ds. \tag{2.26}$$

By similar ideas as above, we get

$$\begin{aligned} \Big|\mathscr{P}_2\Big| & \le |\mu-\mu'|\int_0^t s^{\alpha-1}s^{1-\alpha}|z_{\mu'}(s)|ds \\ & \le 2e^{\frac{2T\mu KT^\alpha}{\alpha}}|y_1||\mu-\mu'|\int_0^t ds \le 2Te^{\frac{2T\mu KT^\alpha}{\alpha}}|y_1||\mu-\mu'| \end{aligned} \tag{2.27}$$

where we have used (2.5). Combining (2.25), (2.26) and (2.27), we derive that

$$\Big|z_\mu(t) - z_{\mu'}(t)\Big| \le 2Te^{\frac{2T\mu KT^\alpha}{\alpha}}|y_1||\mu-\mu'| + K\mu\int_0^t \Big|z_\mu(s) - z_{\mu'}(s)\Big|ds.$$

In view of Gronwall's inequality, we obtain that

$$\Big|z_\mu(t) - z_{\mu'}(t)\Big| \le 2Te^{\frac{2T\mu KT^\alpha}{\alpha}}e^{K\mu T}|y_1||\mu-\mu'|.$$

It follows from (2.22) that

$$\begin{aligned} \Big|y_\mu(t) - y_{\mu'}(t)\Big| & \le \int_0^t \Big|z_\mu(s) - z_{\mu'}(s)\Big|ds \le 2Te^{\frac{2T\mu KT^\alpha}{\alpha}}e^{K\mu T}|y_1||\mu-\mu'|\int_0^t ds \\ & \le 2T^2e^{\frac{2T\mu KT^\alpha}{\alpha}}e^{K\mu T}|y_1||\mu-\mu'|. \end{aligned}$$

□

3 Declarations

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Ethical Approval

Not applicable.

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References

[1] H. Afshari, V. Roomi, and M. Nosrati, *Existence and uniqueness for a fractional differential equation involving Atangana–Baleanu derivative by using a new contraction*, Letters in Nonlinear Analysis and its Applications **1** (2023), no. 2, 52–56.

[2] B. Ahmad, M. Alghanmi, A. Alsaedi, and R. V. Agarwal, *On an impulsive hybrid system of conformable fractional differential equations with boundary conditions*, Internat. J. Systems Sci. **51** (2020), no. 5, 958–970.

[3] F. M. Alharbi, D. Baleanu, and A. Ebaid, *Physical properties of the projectile motion using the conformable derivative*, Chinese Journal of Physics **58** (2019), 18–28.

[4] H. Batarfi, J. Losada, J. J. Nieto, and W. Shammakh, *Three-point boundary value problems for conformable fractional differential equations*, J. Funct. Spaces (2015), 1–6, Art. ID 706383.

[5] B. Bayour and D. F. M. Torres, *Existence of solution to a local fractional nonlinear differential equation*, J. Comput. Appl. Math. **312** (2017), 127–133.

[6] A. Benkerrouche, M. Souid, E. Karapınar, and A. Hakem, *On the boundary value problems of Hadamard fractional differential equations of variable order*, Math. Methods Appl. Sci. **46** (2023), no. 3, 3187–3203.

[7] M. Bouaouid, K. Hilal, and S. Melliani, *Existence of mild solutions for conformable fractional differential equations with nonlocal conditions*, Rocky Mountain J. Math. **50** (2020), no. 3, 871–879.

[8] W. S. Chung, *Fractional Newton mechanics with conformable fractional derivative*, Journal of Computational and Applied Mathematics **290** (2015), 150–158.

[9] Vo Thi Thanh Ha, Hung Ngo, and Phuong Nguyen Duc, *Identifying inverse source for diffusion equation with conformable time derivative by fractional Tikhonov method*, Advances in the Theory of Nonlinear Analysis and its Application **6** (2022), no. 4, 433–450.

[10] S. He, K. Sun, X. Mei, B. Yan, and S. Xu, *Numerical analysis of a fractional-order chaotic system based on conformable fractional-order derivative*, Eur. Phys. J. Plus **132** (2017).

[11] E. Karapinar, H. D. Binh, N. H. Luc, and N. H. Can, *On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems*, Adv. Difference Equ. (2021), Paper No. 70, 24 pp.

[12] E. Karapinar, H. D. Binh, N. H. Luc, and N. H. Can, *On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems*, Advances in Difference Equations **2021** (2021).

[13] E. Karapinar, A. Fulga, M. Rashid, L. Shahid, and H. Aydi, *Large Contractions on Quasi-Metric Spaces with an Application to Nonlinear Fractional Differential-Equations*, Mathematics **7** (2019).

[14] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. **264** (2014), 65–70.

[15] Salim Krim, Abdelkrim Salima, and Mouffak Benchohra, *On implicit Caputo tempered fractional boundary value problems with delay*, Letters in Nonlinear Analysis and its Applications **1** (2023), no. 1, 12–29.

[16] V. F. Morales-Delgado, J. F. Gómez-Aguilar, R. F. Escobar-Jiménez, and M. A. Taneco-Hernández, *Fractional conformable derivatives of Liouville–Caputo type with low-fractionality*, Physica A: Statistical Mechanics and its Applications **503** (2018), 424–438.

[17] W. Qiu, M. Fečkan, D. O'Regan, and J. R. Wang, *Convergence analysis for iterative learning control of conformable impulsive differential equations*, Bull. Iranian Math. Soc. **48** (2022), no. 1, 193–212.

[18] Vahid Roomi, Hojjat Afshari, and Monireh Nosrati, *Existence and uniqueness for a fractional differential equation involving Atangana–Baleanu derivative by using a new contraction*, Letters in Nonlinear Analysis and its Applications **1** (2023), no. 2, 52–56.

[19] A. Souahi, A. Ben Makhlouf, and M. A. Hammami, *Stability analysis of conformable fractional-order nonlinear systems*, Indag. Math. **28** (2017), 1265–1274.

[20] N. H. Tuan, N. V. Tien, D. O'Regan, N. H. Can, and N. V. Thinh, *New results on continuity by order of derivative for conformable parabolic equations*, Fractals **31** (2023), no. 4.

[21] N. H. Tuan, N. V. Tien, and C. Yang, *On an initial boundary value problem for fractional pseudo-parabolic equation with conformable derivative*, Math. Biosci. Eng. **19** (2022), no. 11, 11232–11259.

[22] Nguyen Hoang Tuan, Nguyen Minh Hai, and Phuong Nguyen Duc, *On the nonlinear Volterra equation with conformable derivative*, Advances in the Theory of Nonlinear Analysis and its Application **7** (2023), no. 2, 292–302.

[23] G. Xiao, J. R. Wang, and D. O'Regan, *Existence and Stability of Solutions to Neutral Conformable Stochastic Functional Differential Equations*, Qualitative Theory of Dynamical Systems **21** (2022).

[24] C. Zhai and Y. Liu, *An integral boundary value problem of conformable integro-differential equations with a parameter*, J. Appl. Anal. Comput. **9** (2019), no. 5, 1872–1883.

[25] W. Zhong and L. Wang, *Basic theory of initial value problems of conformable fractional differential equations*, Adv. Differ. Equ. **2018** (2018), 1–14.

[26] W. Zhong and L. Wang, *Positive solutions of conformable fractional differential equations with integral boundary conditions*, Bound. Value Probl. (2018), Paper No. 136, 12 pp.

[27] H. W. Zhou, S. Yang, and S. Q. Zhang, *Conformable derivative approach to anomalous diffusion*, Physica A: Statistical Mechanics and its Applications **491** (2018), 1001–1013.