Letters on
Applied and Pure Mathematics

On the existence and uniqueness of pseudo almost automorphic solutions for integro differential equations with reflection

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Abstract

In this paper we apply fixed point theory and measure theory to investigate the existence of unique solutions for integro differential equations with reflection (IDE-R). Using almost automorphic functions, we study the solutions of these equations, which are of pseudo almost automorphic ($\mathscr{PAA}$) type, by introducing the Mittag-Lefler function. Finally, we present an example we illustrate the application of the main results obtained.

1 Introduction

In the extensive research on differential equations in the literature, different unique solutions such as periodic, almost periodic, and automorphic have been obtained for these equations [3, 12, 13] and generalizations and ideas are presented in different fields [1, 10] (also researchers considered weighted pseudo almost periodic functions which is a generalization of pseudo almost periodicity functions).

The main purpose in this paper is to investigate the existence of solutions for the IDE-R, which is defined as follows (considering the continuous functions of $k, \varphi : \mathbb{R}^3 \to \mathbb{R}$ and

$\mathcal{L} : \mathbb{R}^+ \to \mathbb{R}^+$):

$$\psi'(w) = \tau\psi(w) + \kappa\psi(-w) + h(w) + k(w, \psi(\vartheta(w)), \psi(\vartheta(-w))) \tag{1.1}$$

$$+ \int_w^{+\infty} \mathcal{L}(z - w)\varphi(z, \psi(\vartheta(w)), \psi(\vartheta(-w)))dw$$

$$+ \int_{-w}^{+\infty} \mathcal{L}(z + w)\varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z)))dz, \quad w \in \mathbb{R},$$

for $\tau \in \mathbb{R}$, $\kappa \in \mathbb{R}^*$. Because of the importance of IDE-R in different fields, the stability of these equations has been studied; we refer the reader to [5, 8]. Also, Gupta studied the existence of a unique bounded almost periodic solution for certain kinds of differential equation [9] and in addition two equations

$$\psi'(w) = \tau\psi(w) + \kappa\psi(-w) + h(w), \quad \kappa \neq 0, \quad w \in \mathbb{R}, \tag{1.2}$$

and

$$\psi'(w) = \tau\psi(w) + \kappa\psi(-w) + h(w) + k(w, \psi(w), \psi(-w)) \quad \kappa \neq 0, \quad w \in \mathbb{R}, \tag{1.3}$$

were investigated [14] and properties of the solutions which are of weighted pseudo almost periodic type were presented. Also by considering two measures η and μ, the authors studied the solutions of equations (1.2) and (1.3), which are of η-pseudo almost periodic (η-$\mathscr{PAP}$) type and μ-pseudo almost periodic (μ-$\mathscr{PAP}$) [14].

This work is organized as follows. In Section 2 we state all the basic concepts and in Section 3 we prove the existence and uniqueness of $\mathscr{PAA}$-solutions for (1.1) with measure which is more general than those in equations (1.2) and (1.3). Finally, in the last section, we investigate the $\mathscr{PAA}$-solution of IDEs-R as an application of ourl results. It should be remembered that the main idea of this article is taken from the main ideas of [11].

2 Preminilaries

Definition 2.1. *[7] The specific two parametric M-LF is defined as follows*

$$E_{\alpha,\theta}(v) = \sum_{\ell=0}^{\infty} \frac{v^\ell}{\Gamma(\ell\alpha + \theta)}, \quad Re(\alpha) > 0, \theta \in \mathbb{C}.$$

Definition 2.2 ([3]). *For Banach space $\mathcal{Y}$ and every $k \in \mathcal{C}(\mathbb{R}, \mathcal{Y})$, suppose that (z_ℓ) is a real sequence. If there is a sub-sequence (z_{ℓ_k}) such that*

$$\lim_{\ell_k \to \infty} k(r + z_{\ell_k}) = k(r) \text{ and } \lim_{\ell_k \to \infty} h(r - z_{\ell_k}) = k(r),$$

then, k is said to be almost automorphic or $k \in \mathscr{AA}(\mathbb{R}, \mathcal{Y})$, for every $r \in \mathbb{R}$.

Definition 2.3 ([2]). *We consider a ζ-field $\mathscr{Z}$ as type Lebesque of $\mathbb{R}$ and suppose $\mathscr{M}$ is the space of all positive measures on $\mathscr{Z}$. Then $\eta \in \mathscr{M}$ if*

(1) $\eta([\tau, \kappa]) < \infty$, for all $\tau \leq \kappa \in \mathbb{R}$,
(2) $\eta(\mathbb{R}) = +\infty$.

Definition 2.4 ([6]). *Given the Banach space $\mathcal{Y}$ and the positive measure $\eta \in \mathscr{M}$, a function $k : \mathbb{R} \to \mathcal{Y}$ that is bounded continuous is called η-ergodic, $k \in \mathcal{E}(\mathbb{R}, \mathcal{Y}, \eta)$, if*

$$\lim_{\mathsf{s} \to \infty} \frac{1}{\eta([-\mathsf{s}, \mathsf{s}])} \int_{[-\mathsf{s}, \mathsf{s}]} \|k(w)\| d\eta(w) = 0,$$

where

$$\eta([-\mathsf{s}, \mathsf{s}]) := \int_{-\mathsf{s}}^{\mathsf{s}} d\eta(r).$$

Definition 2.5 ([4]). *Given the Banach space $\mathcal{Y}$ and the positive measure $\eta \in \mathcal{M}$, a function $k : \mathbb{R} \to \mathcal{Y}$ that is continuous is called η-$\mathscr{PAA}$ if*

$$k = h_1 + \mathtt{u}_1,$$

where h_1 is an almost automorphic function ($h_1 \in \mathscr{AA}(\mathbb{R}, \mathcal{Y})$) and $\mathtt{u}_1$ is an ergodic function.

To prove the main results of this paper, we consider the following hypotheses:

($\mathcal{N}_1$) There exist a continuous and increasing function $\vartheta : \mathbb{R} \to \mathbb{R}$ such that for all $v \in \mathscr{AA}(\mathbb{R}, \mathbb{R})$, we have $vo\vartheta \in \mathscr{AA}(\mathbb{R}, \mathbb{R})$.

($\mathcal{N}_2$) For every $\gamma \in \mathbb{R}$, there exist $\vartheta > 0$ and a bounded interval J such that for positive measure η, we have $\eta(\{\tau + \gamma : \tau \in \mathscr{U}\}) \le \vartheta\eta(\mathscr{U})$, whenever $\mathscr{U} \in \mathscr{L}$ satisfies $\mathscr{U} \cap J = \emptyset$.

($\mathcal{N}_3$) There exist $\jmath, \ell > 0$ such that for all $\mathscr{U} \in \mathscr{L}$,

$$\eta(-\mathscr{U}) \le \jmath + \ell\eta(\mathscr{U}).$$

($\mathcal{N}_4$) There is a function $\rho : \mathbb{R} \to \mathbb{R}^+$ such that for all $\mathtt{E} \in \mathbb{B}(\mathbb{R})$, $\eta \in \mathcal{M}$ and $\eta_\vartheta(\mathtt{E}) = \eta\left(\vartheta^{-1}(\mathtt{E})\right)$ we have $d\eta_\vartheta(r) \le \rho(r)d\eta(r)$, ρ is also continuous, strictly increasing and

$$\limsup \frac{\eta[-\mathtt{P}(\mathtt{a}), \mathtt{P}(\mathtt{a})]}{\eta[-\mathtt{a}, \mathtt{a}]} \mathtt{Q}(\mathtt{P}(\mathtt{a})) < +\infty,$$

where $\mathtt{P}(\mathtt{a}) = |\vartheta(\mathtt{a})| + |\vartheta(-\mathtt{a})|$ and $\mathtt{Q}(\mathtt{P}(\mathtt{a})) = \sup_{r \in [-\mathtt{P}(\mathtt{a}), \mathtt{P}(\mathtt{a})]\rho(\mathtt{P})}.$

($\mathcal{N}_5$) Given $\rho = \sqrt{\tau^2 - \kappa^2}$, where $\tau > \kappa$, the following holds

$$\mathtt{D}_1(\rho, \eta) := \sup_{\mathtt{s}>0} \left\{ \int_{-\mathtt{s}}^{\mathtt{s}} \sum_{\ell=0}^{\infty} \frac{(-\rho(r + \mathtt{s}))^\ell}{\Gamma(\ell\alpha + \theta)} d\eta(r) \right\} < \infty,$$

and

$$\mathtt{D}_2(\rho, \eta) := \sup_{\mathtt{s}>0} \left\{ \int_{-\mathtt{s}}^{\mathtt{s}} \sum_{\ell=0}^{\infty} \frac{(-\rho(-r + \mathtt{s}))^\ell}{\Gamma(\ell\alpha + \theta)} d\eta(r) \right\} < \infty.$$

($\mathcal{N}_6$) $k : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ has a Lipschitz coefficient $\mathtt{H}_k > 0$ such that

$$|k(r, v_1, w_1) - k(r, v_2, w_2)| \le \mathtt{H}_k(|v_1 - v_2| + |w_1 - w_2|),$$

for all $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$.

($\mathcal{N}_7$) $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ has a Lipschitz coefficient $\mathtt{H}_\varphi > 0$ such that

$$|\varphi(r, \psi_1, \psi_2) - \psi(r, \mathcal{F}_1, \mathcal{F}_2)| < \mathtt{H}_\varphi(|\psi_1 - \mathcal{F}_1| + |\psi_2 - \mathcal{F}_2|),$$

for all $\psi_1, \psi_2, \mathcal{F}_1, \mathcal{F}_2 \in \mathbb{R}$.

($\mathcal{N}_8$) There exists $\mathcal{L} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g = \int_0^{+\infty} \mathcal{L}(w)dw < \infty$.

To prove the results we consider two states for the Lipschitz coefficients of the functions above. In one state (above) these coefficients are constant and in the second state (below) they are not constant. In the following, we express the necessary conditions according to the second state.

($\mathcal{N}_9$) $k : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ has a Lipschitz function $\mathtt{H}_k \in \mathtt{L}^p(\mathbb{R}, \mathbb{R}, dv) \cap \mathtt{L}^p(\mathbb{R}, \mathbb{R}, d\eta)$ such that

$$|k(r, v_1, w_1) - k(r, v_2, w_2)| \le \mathtt{H}_k(r)(|v_1 - v_2| + |w_1 - w_2|),$$

where $\eta \in \mathcal{M}$, $p > 1$ and for all $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$.

$(\boldsymbol{\mathcal{N}_{10}})$ $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ has a Lipschitz function $\mathrm{H}_\varphi \in \mathrm{L}^p(\mathbb{R}, \mathbb{R}, dv) \cap \mathrm{L}^p(\mathbb{R}, \mathbb{R}, d\eta)$ such that

$$|\varphi(r, v_1, w_1) - \varphi(r, v_2, w_2)| \le \mathrm{H}_\varphi(r)(|v_1 - v_2| + |w_1 - w_2|),$$

where $\eta \in \mathscr{M}$, $p > 1$ and for all $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$.

$(\boldsymbol{\mathcal{N}_{11}})$ There exists $\mathcal{L} : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$\int_0^{+\infty} (\mathcal{L}(w))^\gamma dw < +\infty, \text{ for all } \gamma > 1.$$

Theorem 2.1 ([6]). *If we assume that condition $(\boldsymbol{\mathcal{N}_2})$ holds, then $\mathscr{PAA}(\mathbb{R}, \mathcal{Y}, \eta)$ $(\eta - \mathscr{PAA})$ is translation invariant and $(\mathscr{PAA}(\mathbb{R}, \mathcal{Y}, \eta), \|.\|_\infty)$ is a Banach space, where $\eta \in \mathscr{M}$.*

Lemma 2.1 ([14]). *If we assume that condition $(\boldsymbol{\mathcal{N}_3})$ holds, then for $h \in \mathscr{PAA}(\mathbb{R}, \mathcal{Y}, \eta)$, we have*

$$h(-r) \in \mathscr{PAA}(\mathbb{R}, \mathcal{Y}, \eta),$$

for every $r \in \mathbb{R}$.

Lemma 2.2 ([14]). *Suppose $\eta \in \mathscr{M}$ is a measure that holds in condition $(\boldsymbol{\mathcal{N}_2})$, then for all $p \ge 1$,*

$$\mathrm{L}^p(\mathbb{R}, \mathcal{Y}, d\eta) \subset \mathcal{E}(\mathbb{R}, \mathcal{Y}, \eta).$$

Lemma 2.3 ([11]). *For every $F \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$, if we assume that conditions $(\boldsymbol{\mathcal{N}_1})$ and $(\boldsymbol{\mathcal{N}_4})$ are true, then we have*

$$F(\vartheta(r)) \in (\mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)),$$

where $r \in \mathbb{R}$.

Lemma 2.4 ([11]). *For every $k \in \mathscr{PAA}(\mathbb{R}^3, \mathbb{R}, \eta)$ and $F \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$, if we assume that conditions $(\boldsymbol{\mathcal{N}_4}), (\boldsymbol{\mathcal{N}_6})$ and $(\boldsymbol{\mathcal{N}_1})$-$(\boldsymbol{\mathcal{N}_3})$are true, then we have*

$$k(r, F(\vartheta(r)), F(\vartheta(-r)) \in (\mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)),$$

where $r \in \mathbb{R}$.

Lemma 2.5 ([11]). *For every $\varphi \in \mathscr{PAA}(\mathbb{R}^3, \mathbb{R}, \eta)$ and $F \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$, if we assume that conditions $(\boldsymbol{\mathcal{N}_4}), (\boldsymbol{\mathcal{N}_6}), (\boldsymbol{\mathcal{N}_8})$and $(\boldsymbol{\mathcal{N}_1})$-$(\boldsymbol{\mathcal{N}_3})$are true, then we have*

$$\int_r^\infty \mathcal{L}(z - r)\varphi(z, F(\vartheta(z)), F(\vartheta(-z))dz \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta),$$

where $r \in \mathbb{R}$.

Lemma 2.6 ([11]). *For every $h \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}^2, \eta)$, $\varphi \in \mathscr{PAA}(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}, \eta)$, if we assume that conditions $(\boldsymbol{\mathcal{N}_2})$ and $(\boldsymbol{\mathcal{N}_{10}})$ are true, then*

$$\varphi(r, h(r)) \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta),$$

where $r \in \mathbb{R}$ and $\eta \in \mathscr{M}$.

3   Existence of a unique η-$\mathscr{PAA}$ solution for equation (1.1) in two-states

Theorem 3.1. *Let $k, \varphi \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}, \eta)$ and assume that $(\mathcal{N_4})$-$(\mathcal{N_8})$, $(\mathcal{N_1})$-$(\mathcal{N_3})$ are satisfied. Then equation (1.1) has a unique η-$\mathscr{PAA}$ solution if*

$$\frac{|\rho - \tau| + |\rho + \tau| + 2|\kappa|}{\rho\left(\sum_{\ell=0}^{\infty} \frac{(\rho r)^\ell}{\Gamma(\ell\alpha+\theta)} \sum_{\ell=0}^{\infty} \frac{(-\rho)^\ell r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha+\theta)}\right)}(\mathrm{H}_k + 2g\mathrm{H}_\varphi) < 1.$$

Proof. For any $k, \varphi \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$, we consider the following equation:

$$\Xi v(r) = -\frac{1}{2\rho}\left[\sum_{\ell=0}^{\infty} \frac{(\rho r)^\ell}{\Gamma(\ell\alpha+\theta)} \int_r^\infty \sum_{\ell=0}^{\infty} \frac{(-\rho w)^\ell}{\Gamma(\ell\alpha+\theta)}((\rho-\tau)k(w, v(w), v(-w)) + \kappa k(-w, v(-w), v(w)))dw\right]$$

(3.4)

$$+\frac{1}{2\rho}\left[\sum_{\ell=0}^{\infty} \frac{(-\rho r)^\ell}{\Gamma(\ell\alpha+\theta)} \int_{-\infty}^r \sum_{\ell=0}^{\infty} \frac{(\rho w)^\ell}{\Gamma(\ell\alpha+\theta)}((\rho+\tau)k(w, v(w), v(-w)) - \kappa k(-w, v(-w), v(w)))dw\right]$$

$$-\frac{1}{2\rho}\left[\sum_{\ell=0}^{\infty} \frac{(\rho r)^\ell}{\Gamma(\ell\alpha+\theta)} \int_r^\infty \sum_{\ell=0}^{\infty} \frac{(-\rho w)^\ell}{\Gamma(\ell\alpha+\theta)}((\rho-\tau)h(w) + \kappa h(-w))dw\right]$$

$$+\frac{1}{2\rho}\left[\sum_{\ell=0}^{\infty} \frac{(-\rho r)^\ell}{\Gamma(\ell\alpha+\theta)} \int_{-\infty}^r \sum_{\ell=0}^{\infty} \frac{(\rho w)^\ell}{\Gamma(\ell\alpha+\theta)}((\rho+\tau)h(w) - \kappa h(-w))dw\right],$$

where

$$h(w) = \int_w^{+\infty} \mathcal{L}(z-w)\varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z)))dz$$

$$+ \int_{-w}^{+\infty} \mathcal{L}(z+w)\varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z)))dz.$$

Now, considering Lemmas 2.1, 2.4 and 2.5 and for each $w \in \mathbb{R}$, we have $\int_w^{+\infty} \mathcal{L}(z-w)\varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z)))dz \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$, and $\int_{-w}^{+\infty} \mathcal{L}(z+w)\varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z)))dz \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$ so this means that $h \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$. Thus, according to the basic lemmas and theorems presented in the previous section, Ξ is a mapping from $\mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$ to $\mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$. Now, we consider the following equation

$$\mathcal{G}(r, F(\vartheta(r)), F(\vartheta(-r))) = k(r, F(\vartheta(r)), F(\vartheta(-r))) \qquad (3.5)$$

$$+ \int_w^{+\infty} \mathcal{L}(z-w)\varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z)))dz$$

$$+ \int_{-w}^{+\infty} \mathcal{L}(z+w)\varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z)))dz.$$

We will show that $\Xi : \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta) \to \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$ is a contraction. According to $(\mathcal{N_1})$,

it follows that $\vartheta : \mathbb{R} \to \mathbb{R}$ is bijective. Then, we have

$$
\begin{aligned}
| \mathcal{G}(r, & F(\vartheta(r)), F(\vartheta(-r))) - \mathcal{G}(r, \psi(\vartheta(r)), \psi(\vartheta(-r))) | \\
= & |k(r, F(\vartheta(r)), F(\vartheta(-r))) - k(r, \psi(\vartheta(r)), \psi(\vartheta(-r)))| \\
& + \int_{r}^{+\infty} \mathcal{L}(z - r)(\varphi(z, F(\vartheta(z)), F(\vartheta(-z))) - \varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z))))dz \\
& + \int_{-r}^{+\infty} \mathcal{L}(r + z)(\varphi(z, F(\vartheta(z)), F(\vartheta(-z))) - \varphi(z, \psi(\vartheta(z)), \psi(\vartheta(-z))))dz \\
\leq & |k(r, F(\vartheta(r)), F(\vartheta(-r))) - k(r, \psi(\vartheta(r)), \psi(\vartheta(-r)))| \\
& + \int_{0}^{+\infty} \mathcal{L}(z)(\varphi((z + r), F(\vartheta(z + r)), F(\vartheta(-(z + r)))) - \varphi((z + r, \psi(\vartheta(z + r)), \psi(\vartheta(-(z + r))))))dz \\
& + \int_{0}^{+\infty} \mathcal{L}(z)(\varphi(z - r, F(\varphi(z - r)), F(\vartheta(-(z - r)))) - \varphi(z - r, \psi(\vartheta(z - r)), \psi(\vartheta(-(z - r))))))dr \\
\leq & 2 \left(\mathrm{H}_k + 2g\mathrm{H}_\varphi \right) \|F - \psi\|_\infty,
\end{aligned}
$$

for all $\psi, F \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$. Therefore

$$
|\Xi F(r) - \Xi \psi(r)| \leq \frac{|\rho - \tau| + |\rho + \tau| + 2|\kappa|}{\rho(\sum_{\ell=0}^\infty \frac{(\rho r)^\ell}{\Gamma(\ell\alpha + \theta)} \sum_{\ell=0}^\infty \frac{(-\rho)^\ell r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha + \theta)})}(\mathrm{H}_k + 2g\mathrm{H}_\varphi)\|F - \psi\|_\infty.
$$

Since

$$
\frac{|\rho - \tau| + |\rho + \tau| + 2|\kappa|}{\rho(\sum_{\ell=0}^\infty \frac{(\rho t)^\ell}{\Gamma(\ell\alpha + \theta)} \sum_{\ell=0}^\infty \frac{(-\rho)^\ell r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha + \theta)})}(\mathrm{H}_k + 2g\mathrm{H}_\varphi)\|F - \psi\|_\infty < 1,
$$

we have that $\Xi : \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta) \to \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$ is a contraction and according to the Banach fixed point theorem Ξ has a unique fixed point in $\mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$. We conclude that (1.1) has a unique η-$\mathscr{PAA}$ solution. $\qquad\square$

Theorem 3.2. *Consider $k, \varphi \in \mathscr{PAP}(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}, \eta)$ and assume that conditions ($\mathcal{N}_1$)-($\mathcal{N}_5$) and ($\mathcal{N}_9$)-($\mathcal{N}_{10}$) are satisfied. Then, equation (1.1) has a unique η-$\mathscr{PAA}$ solution if*

$$
\|\mathrm{H}_k\|_{L^p(\mathbb{R}, \mathbb{R}, dv)} + 2 \left(\int_0^{+\infty} (\mathcal{L}(w))^q \right)^{\frac{1}{q}} \|\mathrm{H}_\varphi\|_{L^p(\mathbb{R}, \mathbb{R}, dv)} < \frac{\sum_{\ell=0}^\infty \frac{((-\rho)^{\frac{1}{q}} r)^\ell}{\rho q^{\frac{1}{q}} \Gamma(\ell\alpha + \theta)} \sum_{\ell=0}^\infty \frac{(-\rho)^{\frac{\ell}{q}} r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha + \theta)}}{|\rho - \tau| + |\rho + \tau| + 2|\kappa|},
$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. (Here we are considering Lipschitz functions). Similar to the proof of the above theorem, we consider the Ξ function. As we have proved, Ξ is a mapping from $\mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$ to $\mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$. Now to complete the proof of this theorem we show that Ξ is a contraction. In the following, we use equation (3.5) introduced in the previous theorem. Then, we have

$$
\begin{aligned}
| \mathcal{G}(r, & F(\vartheta(r)), F(\vartheta(-r))) - \mathcal{G}(r, \psi(\vartheta(r)), \psi(\vartheta(-r))) | \\
\leq & |k(r, F(\vartheta(r)), F(\vartheta(-r))) - k(r, \psi(\vartheta(r)), \psi(\vartheta(-r)))| \\
& + \int_{0}^{+\infty} \mathcal{L}(z)(\varphi((z + r), F(\vartheta(z + r)), F(\vartheta(-(z + r)))) - \varphi((z + r, \psi(\vartheta(z + r)), \psi(\vartheta(-(z + r))))))dz \\
& + \int_{0}^{+\infty} \mathcal{L}(z)(\varphi(z - r, F(\vartheta(z - r)), F(\vartheta(-(z - r)))) - \varphi(z - r, \psi(\vartheta(z - r)), \psi(\vartheta(-(z - r))))))dz \\
\leq & \left[2\mathrm{H}_k(r) + 4 \left(\int_0^{+\infty} (\mathcal{L}(w))^q dk \right)^{\frac{1}{q}} \|\mathrm{H}_\varphi\|_{L^p(\mathbb{R}, \mathbb{R}, dv)} \right] \|F - \psi\|_\infty,
\end{aligned}
$$

where $\psi, F \in \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$, and hence

$$|\Xi F(r) - \Xi \psi(r)| \leq \frac{|\rho - \tau| + |\rho + \tau| + 2|\kappa|}{\sum_{\ell=0}^{\infty} \frac{((-\rho)^{\frac{1}{q}} r)^{\ell}}{\rho q^{\frac{1}{q}} \Gamma(\ell\alpha+\theta)} \sum_{\ell=0}^{\infty} \frac{(-\rho)^{\frac{\ell}{q}} r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha+\theta)}} \left[\|\mathrm{H}_k\|_{L^p(\mathbb{R}, \mathbb{R}, dv)} \right.$$

$$\left. + 2 \left(\int_0^{+\infty} (\mathcal{L}(w))^q dw \right)^{\frac{1}{q}} \|\mathrm{H}_{\varphi}\|_{L^p(\mathbb{R}, \mathbb{R}, dv)} \right] \|F - \psi\|_{\infty}.$$

Since

$$\frac{|\rho - \tau| + |\rho + \tau| + 2|\kappa|}{\sum_{\ell=0}^{\infty} \frac{((-\rho)^{\frac{1}{q}} r)^{\ell}}{\rho q^{\frac{1}{q}} \Gamma(\ell\alpha+\theta)} \sum_{\ell=0}^{\infty} \frac{(-\rho)^{\frac{\ell}{q}} r^{\ell+1}}{(\ell+1)\Gamma(\ell\alpha+\theta)}} \left[\|\mathrm{H}_k\|_{L^p(\mathbb{R}, \mathbb{R}, dv)} + 2 \left(\int_0^{+\infty} (\mathcal{L}(w))^q dw \right)^{\frac{1}{q}} \|\mathrm{H}_{\varphi}\|_{L^p(\mathbb{R}, \mathbb{R}, dv)} \right] < 1,$$

then the operator $\Xi : \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta) \to \mathscr{PAA}(\mathbb{R}, \mathbb{R}, \eta)$ is a contraction and according to the Banach fixed point theorem Ξ has a unique fixed point in $\mathscr{PAP}(\mathbb{R}, \mathbb{R}, \eta)$. We conclude that equation (1.1) has a unique η-$\mathscr{PAA}$ solution. $\qquad\square$

Proposition 3.1. *If we use the exponential function instead of Mittag-Leffler function, all the conditions and theorems raised are also valid, refer to [11] for details.*

4 An Application

We consider a measure that is defined as follows

$$d\eta(r) = \varpi(r)dr,$$

where $\varpi(r) = \sum_{\ell=0}^{\infty} \frac{(\sin r)^{\ell}}{\Gamma(\alpha\ell+\theta)}$ and $r \in \mathbb{R}$. According to the measure we have defined $\eta \in \mathscr{M}$, then condition $(\mathcal{N}_2)$ is true. Let $J = [\tau, \kappa]$, Given that $2 + \sin r \geq \sin(-r)$, we have

$$\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\Gamma(\alpha\ell+\theta)} dr \leq d\eta(r) = \sum_{\ell=0}^{\infty} \frac{(\sin r)^{\ell}}{\Gamma(\alpha\ell+\theta)} dr \leq \sum_{\ell=0}^{\infty} \frac{1}{\Gamma(\alpha\ell+\theta)} dr \implies \qquad (4.6)$$

$$\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\Gamma(\alpha\ell+\theta)} (\kappa - \tau) \leq \eta(r) \leq \sum_{\ell=0}^{\infty} \frac{1}{\Gamma(\alpha\ell+\theta}) (\kappa - \tau) \implies$$

$$\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\Gamma(\alpha\ell+\theta)} (\kappa - \tau) \leq \eta(r) \leq \sum_{\ell=0}^{\infty} \frac{1}{\Gamma(\alpha\ell+\theta)} (\kappa - \tau) \implies$$

$$1 + \sum_{\ell=0}^{\infty} \frac{2^{\ell}}{\Gamma(\alpha\ell+\theta)} (\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\Gamma(\alpha\ell+\theta)} (\kappa - \tau)) \leq 1 + \sum_{\ell=0}^{\infty} \frac{2^{\ell}}{\Gamma(\alpha\ell+\theta)} \eta(r) \leq$$

$$1 + \sum_{\ell=0}^{\infty} \frac{2^{\ell}}{\Gamma(\alpha\ell+\theta)} (\sum_{\ell=0}^{\infty} \frac{1}{\Gamma(\alpha\ell+\theta)} (\kappa - \tau)),$$

then, $1 + \sum_{\ell=0}^{\infty} \frac{2^{\ell}}{\Gamma(\alpha\ell+\theta)} \eta(J) \geq \eta(-J)$. Therefore, we conclude that condition $(\mathcal{N}_3)$ holds. Given the main differential integral equation of this paper, i.e. equation(1.1), we consider the following

equation

$$v'(r) = 4v(r) + \sqrt{7}v(-r) + \frac{\sum_{\ell=0}^{\infty} \frac{(-|r|)^\ell}{\Gamma(\alpha\ell+\theta)}}{15} [\sin v(\frac{1}{r+1} - e) + \cos v(-(\frac{1}{r+1}) + e)] \quad (4.7)$$

$$+ \int_r^{+\infty} \mathcal{L}(z-r) \frac{\sum_{\ell=0}^{\infty} \frac{(-|z|)^\ell}{\Gamma(\alpha\ell+\theta)}}{15} [\sin v(\frac{1}{z+1} - e) + \cos v(-(\frac{1}{z+1}) + e)]dz$$

$$+ \int_{-r}^{+\infty} \mathcal{L}(z+r) \frac{\sum_{\ell=0}^{\infty} \frac{(-|z|)^\ell}{\Gamma(\alpha\ell+\theta)}}{15} [\sin v(\frac{1}{z+1} - e) + \cos v(-(\frac{1}{z+1}) + e)]dz,$$

where e represents the delay and a positive real number, $\mathcal{L}(z) = \sum_{\ell=0}^{\infty} \frac{(-|z|)^\ell}{\Gamma(\alpha\ell+\theta)}$ for all $z \in \mathbb{R}^+$. In order for condition $(\mathcal{N_1})$ to be true, we set $\vartheta(r) = \frac{1}{r+1} - e$. Next, we put

$$\tau = 4, \kappa = \sqrt{7}, \rho = \sqrt{\tau^2 - \kappa^2} = 3 \text{ and } k(r, v, w) = \varphi(r, v, w)$$

$$= \frac{1}{15} \sum_{\ell=0}^{\infty} \frac{(-|r|)^\ell}{\Gamma(\alpha\ell+\theta)} [\sin v + \cos w].$$

Let $p = q = \frac{1}{2}$. Then

$$|k(r, v_1, w_1) - k(r, v_2, w_2)| \le \mathrm{H}_k(r) (|v_1 - v_2| + |y_1 - w_2|),$$

for all $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$, and

$$|\varphi(r, v_1, w_1) - \varphi(r, v_2, w_2)| \le \mathrm{H}_\varphi(r) (|v_1 - v_2| + |w_1 - w_2|),$$

for all $(v_1, w_1), (v_2, w_2) \in \mathbb{R}^2$, where

$$\left[r \to \mathrm{H}_k(r) = \mathrm{H}_\varphi(r) = \frac{1}{15} \sum_{\ell=0}^{\infty} \frac{(-|r|)^\ell}{\Gamma(\alpha\ell+\theta)} \right] \in L^2(\mathbb{R}, \mathbb{R}, dv) \cap L^2(\mathbb{R}, \mathbb{R}, d\eta).$$

We consider $\alpha = 1, \theta = 1$, and

$$\|\mathrm{H}_k\|_{L^2(\mathbb{R},\mathbb{R},dv)} = \|\mathrm{H}_\varphi\|_{L^2(\mathbb{R},\mathbb{R},dv)} = \frac{1}{15} \text{ and } \|\mathrm{H}_k\|_{L^2(\mathbb{R},\mathbb{R},d\eta)} = \|\mathrm{H}_\varphi\|_{L^2(\mathbb{R},\mathbb{R},d\eta)} \le \frac{1}{15}\sqrt{\exp(1)}.$$

This implies that hypothesis $(\mathcal{N_7})$ is satisfied. Note

$$\|\mathrm{H}_k\|_{L^2(\mathbb{R},\mathbb{R},dv)} + 2 \left(\int_0^{+\infty} (\mathcal{L}(w))^2 dw \right)^{\frac{1}{2}} \|\mathrm{H}_\varphi\|_{L^2(\mathbb{R},\mathbb{R},dv)}$$

$$= \frac{\sqrt{2}+1}{15} < \frac{e^{18}}{|\rho - \tau| + |\rho + \tau| + 2|\kappa|} = \frac{e^{18}}{8 + 2\sqrt{7}}.$$

Thus, all the conditions of theorem 3.2 are satisfied. Then equation (4.7) has a unique $\eta\text{-}\mathscr{PAP}$ solution.

5 Conclusion

Because of the importance of differential integral equations with reflection and delay (IDE-RD), many researchers have studied these equations. For example, solutions were established of the main equation when $\vartheta(r) = r$, while in this paper $\vartheta(r) \ne r$ is considered. In this work, by considering the Banach FPT, properties of almost automorphic functions, positive measurements, and using the M-LF, we establish solutions of the IDE-RD. Two cases on the existence of a unique solutions of this equation which are of $\mathscr{PAA}$ type were discussed. Finally, we presented a practical example of the results obtained.

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