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Letters on
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# On approximation and error estimate for Poisson equation in Banach space

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## Abstract

This article investigates the Poisson equation inverse source problem. This is an ill-posed, i.e. a small change in the data will lead to a very large change in the solution. Therefore, a regularized solution is necessary. In this work, we construct the regularized solution by truncation method. We also investigate the convergent rate between the regularized solution and the sought solution in $L^j(0, \pi)$.

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## 1 Introduction

Inverse source problem is of great importance in many applications, such as heat source determination, and heat conduction problem, see in [1, 4]. We have also a variety of research on inverse source problems in the Poisson equation adopted numerical methods, see in [16, 23, 25]. Applications in science and engineering rely heavily on inverse source problems (see [6, 12]). Finding the hidden source's location, size, and shape from the boundary's measured data is the inverse source problem of the Poisson equation's source term determination problem. The nonlinearity and illusory nature of this inverse source problem can be seen in the fact that, even if a solution is found, it does not always rely on the measured data. The given data could have small errors that result in large errors in the solution. As a result, our proposed algorithms should make use of regularization methods. Numerous studies have been conducted on the Poisson equation's inverse source problems [2, 3, 8, 14–16]. Bubnov, Erokhin and Isakov [3, 17] introduced a few hypothetical outcomes to recreate the obscure source or impediments from over-determined limit estimations of arrangements of the Poisson condition. From over-determined boundary measurements of Laplace equation solutions, the inverse problem of locating pointwise or small-size conductivity defaults in a plane domain was solved by [2]. Several efficient numerical algorithms for resolving Poisson equation inverse source problems were proposed [16, 21]. In order to identify hidden obstacles, Hanke and Rundell used the rational approximation method, see in [13]. Iterative algorithms exist to obtain source parameters from boundary measurement data [14, 15, 18, 24, 26]. An inverse potential problem for reconstructing an obstacle's shape was solved using iterative algorithms in Hettlich and Rundell [15]. In this paper, with $\Omega = (0, \pi)$, we consider to find a pair of functions $\big(\nu(x, y), f(x)\big)$ satisfying

$$-\nu_{xx} - \nu_{yy} = f(x), \;\; 0 < x < \pi, \quad 0 < y < +\infty, \tag{1.1}$$

with boundary condition

$$\nu(0, y) = \nu(\pi, y) = 0, \;\; 0 < y < +\infty, \tag{1.2}$$

and final condition

$$\nu(x, 1) = \rho(x), \;\; 0 < x < \pi, \tag{1.3}$$

and

$$\nu(x, 0) = 0, \, \nu(x, y)|_{y\to\infty} \text{ bounded}, \;\; 0 < x < \pi, \tag{1.4}$$

where $\nu(x, 1) = \rho(x)$ is the supplementary condition and $f(x)$ is the unknown source that is only dependent on one spatial variable. The inverse problem of identifying an unknown source is the name given to this issue. Numerous studies have been conducted on the various heat source types cited in the heat equation [5, 10, 19, 32, 33, 37]. We are aware of only a few papers that attempted to identify the random noise-based unknown source in the Poisson equation. Since we cannot anticipate that the measured data function $\rho_\epsilon(x)$ will decay at the same rate in $L^2(0, \pi)$. The ill-posed problem will be addressed by means of the truncation regularization approach in the following section. The truncation regularization strategies have been read up for reverse issues, with the end goal that : Eldén, see [9], Xiong, see [31], Fu, see [11], and Qian, see [28]. In [30], we consider an inverse source problem for Poisson equations in a band domain. It is to determine the source term in the Poisson Equation from a noisy boundary data. The authors propose a modified quasi-reversibility regularization method to solve the inverse source problem and achieve the convergence rate using a priori regularization parameter selection rule. Besides, they also provide numerical examples to check the results achieved in the theoretical proof. There are many methods to regularize this problem (1.1). Readers can see in the following documents [7, 20, 22, 27, 34–36, 38].

The outline of the paper is as follows. In Section 2, we present some preparatory knowledge. In Section 3, we have the solution of problem (1.1). Finally, the main result is in Section 3.

## 2 Preminilaries

For $s > 0$, let $\mathbb{X}^s(0, \pi)$ be the arrangement of all capabilities $f \in L^2(0, \pi)$ with

$$\|f\|^2_{\mathbb{X}^s(0,\pi)} = \sum_{n=1}^{\infty} n^{2s} |\langle f, e_n\rangle|^2 < \infty. \tag{2.5}$$

**Lemma 2.1.** *(See in [36]) If $n \geqslant 1$, we have*

$$\frac{1}{1 - \exp(-n)} \leq 2. \tag{2.6}$$

**Lemma 2.2.** *(See [29]) The following statement are true:*

$$\left.\begin{array}{llll} L^p(0,\pi) \hookrightarrow \mathbb{X}^\mu(0,\pi), & \textit{if} & -\dfrac{N}{4} < \mu \leq 0, & p \geq \dfrac{2N}{N-4\mu}, \\ \mathbb{X}^\mu(0,\pi) \hookrightarrow L^p(0,\pi), & \textit{if} & 0 \leq \mu < \dfrac{N}{4}, & p \leq \dfrac{2N}{N-4\mu}. \end{array}\right\} \tag{2.7}$$

## 3 The solution of problem (1.1)

Generally, the input data $g(x)$ with a noise level $\epsilon$ is merely measured in $L^2(0,\pi)$, and we give that

$$\|\rho - \rho_\epsilon\|_{L^2(0,\pi)} \leq \epsilon. \tag{3.8}$$

We obtain that the solution of problem (1.1) as follows:

$$\nu(x,y) = \sum_{n=1}^{\infty} \frac{1-e^{-ny}}{n^2} \langle f, e_n \rangle \, e_n(x), \tag{3.9}$$

whereby

$$\left\{ e_n = \sqrt{\frac{2}{\pi}} \sin nx, \quad (n = 1, 2, \ldots) \right\}, \;\; \langle f, e_n \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi f(x) \sin nx dx. \tag{3.10}$$

Defining the operator $\mathcal{K} : f \to g$, then we have

$$\rho(x) = \mathcal{K} f(x) = \sum_{n=1}^{+\infty} \langle \rho, e_n \rangle \, e_n = \sum_{n=1}^{\infty} \frac{1-e^{-n}}{n^2} \langle f, e_n \rangle e_n(x) \cdot \tag{3.11}$$

The singular values $\{\varsigma_n\}_{n=1}^{\infty}$ of $\mathcal{K}$ satisfy

$$\varsigma_n = \frac{1-e^{-n}}{n^2}, \tag{3.12}$$

correspondingly

$$\langle \rho, e_n \rangle = \frac{1-e^{-n}}{n^2} \langle f, e_n \rangle \langle e_n, e_n \rangle \,, \tag{3.13}$$

this leads to

$$\langle f, e_n \rangle = \varsigma_n^{-1} \langle \rho, e_n \rangle \,, \tag{3.14}$$

then

$$f(x) = \mathcal{K}^{-1} \rho(x) = \sum_{n=1}^{+\infty} \frac{1}{\varsigma_n} \langle \rho, e_n \rangle \, e_n = \sum_{n=1}^{+\infty} \frac{n^2}{1-e^{-n}} \langle \rho, e_n \rangle \, e_n. \tag{3.15}$$

## 4 Regularization of problem (1.1)

**Theorem 4.1.** *Let us take $\rho_\varepsilon \in L^j(0,\pi)$ such that*

$$\|\rho_\epsilon - \rho\|_{L^j(0,\pi)} \leq \epsilon. \tag{4.16}$$

*Assume that* $f \in \mathbb{X}^{j+s}(0,\pi)$ *for* $s>0$ *and* $0<j<\frac{N}{4}$. *Fourier truncation method help us to establish the regularized solution*

$$f_\epsilon^{\mathcal{B}_\epsilon}(x) = \sum_{n=1}^{\mathcal{B}_\epsilon} \frac{n^2}{1-e^{-n}} \big\langle \rho_\epsilon, e_n \big\rangle e_n(x). \tag{4.17}$$

*Then the error estimate as follow*

$$\Big\| f_\epsilon^{\mathcal{B}_\epsilon} - f \Big\|_{L^{\frac{2N}{N-4j}}(0,\pi)} \lesssim 2\sqrt{\frac{2}{\pi}} C(N,j) \big(\mathcal{B}_\epsilon\big)^{2+s+\frac{N}{j}-\frac{N}{2}} \epsilon + \sqrt{\frac{2}{\pi}} \, |\mathcal{B}_\epsilon|^{-j} \quad \|f\|_{\mathbb{X}^{j+s}(0,\pi)}. \tag{4.18}$$

*By choosing* $\mathcal{B}_\epsilon$ *satisfies that*

$$\lim_{\epsilon \to 0} \left(\mathcal{B}_\epsilon\right)^{2+s+\frac{N}{j}-\frac{N}{2}} \epsilon = 0, \quad \lim_{\epsilon\to 0} \mathcal{B}_\epsilon = +\infty. \tag{4.19}$$

**Remark 4.1.** $\mathcal{B}_\epsilon$ *is chosen as follows:*

$$\mathcal{B}_\epsilon = \epsilon^{\frac{\kappa-1}{s+2+\frac{N}{j}-\frac{N}{2}}}, \quad 0<\kappa<1.$$

*Proof.* First of all, we have

$$f(x) = \sum_{n=1}^{\infty} \frac{n^2}{1-e^{-n}} \langle \rho, e_n \rangle \, e_n = \sum_{n=1}^{\mathcal{B}_\epsilon} \frac{n^2}{1-e^{-n}} \langle \rho, e_n \rangle \, e_n + \sum_{n=\mathcal{B}_\epsilon+1}^{\infty} \frac{n^2}{1-e^{-n}} \langle \rho, e_n \rangle \, e_n \tag{4.20}$$

From (4.17) and (4.20), we get

$$\Big\| f - f_\epsilon^{\mathcal{B}_\epsilon} \Big\|_{\mathbb{X}^s(0,\pi)} = \underbrace{\Big\| \sum_{n=1}^{\mathcal{B}_\epsilon} \frac{n^2}{1-e^{-n}} \big\langle \rho-\rho_\epsilon, e_n \big\rangle e_n(x) \Big\|_{\mathbb{X}^s(0,\pi)}}_{\mathcal{E}_1} + \underbrace{\Big\| \sum_{n=\mathcal{B}_\epsilon+1}^{\infty} \frac{n^2}{1-e^{-n}} \langle \rho, e_n \rangle \, e_n \Big\|_{\mathbb{X}^s(0,\pi)}}_{\mathcal{E}_2}. \tag{4.21}$$

From (4.21), we have estimate of $\mathcal{E}_1$ and $\mathcal{E}_2$ as follows:

**Step 1.** Estimate of $\mathcal{E}_1^2$, using the Lemma 2.1, we know that

$$\begin{aligned}
\mathcal{E}_1^2 &\le \frac{8}{\pi} \sum_{n=1}^{\mathcal{B}_\epsilon} n^{4+2s} \Big| \Big\langle \rho-\rho_\epsilon, e_n \Big\rangle \Big|^2 \\
&\le \frac{8}{\pi} \sum_{n=1}^{\mathcal{B}_\epsilon} n^{4+2s+\frac{2N}{j}-N} n^{\frac{Nj-2N}{j}} \Big| \Big\langle \rho-\rho_\epsilon, e_n \Big\rangle \Big|^2 \\
&\le \frac{8}{\pi} \big(\mathcal{B}_\epsilon\big)^{4+2s+\frac{2N}{j}-N} \sum_{n=1}^{+\infty} n^{\frac{Nj-2N}{j}} \Big\| \rho-\rho_\epsilon \Big\|^2_{L^2(0,\pi)} \\
&\le \frac{8}{\pi} \big(\mathcal{B}_\epsilon\big)^{4+2s+\frac{2N}{j}-N} \Big\| \rho-\rho_\epsilon \Big\|^2_{\mathbb{X}^{\frac{Nj-2N}{2j}}(0,\pi)}.
\end{aligned} \tag{4.22}$$

Since $1<j<2$, we know that $L^j(0,\pi) \hookrightarrow \mathbb{X}^{\frac{Nj-2N}{2j}}(0,\pi)$. Hence, we get

$$\Big\| \rho-\rho_\epsilon \Big\|^2_{\mathbb{X}^{\frac{Nj-2N}{2j}}(0,\pi)} \lesssim C(N,j) \Big\| \rho-\rho_\epsilon \Big\|^2_{L^j(0,\pi)} \le C^2(N,j)\epsilon^2. \tag{4.23}$$

From (4.22), we have

$$\mathcal{E}_1 \le 2\sqrt{\frac{2}{\pi}} C(N,j)\big(\mathcal{B}_\epsilon\big)^{2+s+\frac{N}{j}-\frac{N}{2}}\epsilon. \tag{4.24}$$

**Step 2.** Estimate of $\mathcal{E}_2$, throught the Parseval equality, the norm on $\mathbb{X}^s(0,\pi)$ is calculated as follow

$$\mathcal{E}_2^2 = \frac{2}{\pi}\sum_{n=\mathcal{B}_\epsilon+1}^{\infty} n^{2s}\Big|\Big\langle f, e_n\Big\rangle\Big|^2 = \frac{2}{\pi}\sum_{n=\mathcal{B}_\epsilon+1}^{\infty} n^{-2j}n^{2j+2s}\Big|\Big\langle f, e_n\Big\rangle\Big|^2.$$

It is easy to see that $n^{-2j} \le |\mathcal{B}_\epsilon|^{-2j}$ if $n > \mathcal{B}_\epsilon$ and $j > 0$. Therefore, we get that

$$\mathcal{E}_2^2 \le \frac{2}{\pi}\big|\mathcal{B}_\epsilon\big|^{-2j}\sum_{n=\mathcal{B}_\epsilon+1}^{+\infty} n^{2j+2s}\Big|\Big\langle f, e_n\Big\rangle\Big|^2 = \frac{2}{\pi}\big|\mathcal{B}_\epsilon\big|^{-2j}\big\|f\big\|^2_{\mathbb{X}^{j+s}(0,\pi)}. \tag{4.25}$$

It gives that

$$\mathcal{E}_2 \le \sqrt{\frac{2}{\pi}}\ |\mathcal{B}_\epsilon|^{-j}\big\|f\big\|_{\mathbb{X}^{j+s}(0,\pi)}. \tag{4.26}$$

From two steps, we can conclude that

$$\big\|f - f_\epsilon^{\mathcal{B}_\epsilon}\big\|_{\mathbb{X}^s(0,\pi)} \le 2\sqrt{\frac{2}{\pi}} C(N,j)\big(\mathcal{B}_\epsilon\big)^{2+s+\frac{N}{j}-\frac{N}{2}}\epsilon + \sqrt{\frac{2}{\pi}}\ |\mathcal{B}_\epsilon|^{-j}\big\|f\big\|_{\mathbb{X}^{j+s}(0,\pi)}. \tag{4.27}$$

By using Lemma 2.2, since $0 < j < \frac{N}{4}$, one has $\mathbb{X}^j(0,\pi) \hookrightarrow L^{\frac{2N}{N-4j}}(0,\pi)$, which yields to the desired result (4.18). This completes the proof. □

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