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Existence of nontrivial solutions for the Schrödinger-Poisson system with sign-changing potential

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Abstract

In this article, we are interested in considering the Schrödinger-Poisson system while the potential function V is indefinite, and the negative space of the Schrödinger operator $-\Delta + V$ is finite-dimensional. Different from many other articles, we consider the condition of nonlinearity $g(x,t)$ to be weaker than the Ambrosetti-Rabinowitz condition. The Schrödinger-Poisson system has nontrivial solutions, which can be found through the application of the Local linking theorem.

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1 Introduction

We consider a particular class of the Schrödinger-Poisson system

$$\begin{cases} -\Delta t + V(x)t + \phi t = g(x,t) + k(x), & \text{in } \mathbb{R}^N, \\ -\Delta\phi = t^2, & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

and as $|x| \to \infty$, $t(x) \to 0$, where $N \geq 3$, $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, the nonlinearity $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and $k \in L^2(\mathbb{R}^N)$ satisfy some further conditions. In fact, when the interaction between the unknown electromagnetic field and the nonlinear Schrödinger equation is studied, this system (1) will be generated, see [5] for details.

Note that many papers have studied the case of $k(x) \equiv 0$, and most of them have used the Symmetric Mountain Pass Theorem in [3] or the Fountain Theorem in [4] to prove the existence of both the infinitely many high energy solutions and the ground state solution (for details, see e.g. [1, 2, 7, 14] and its related references). Since the functional of the Schrödinger-Poisson system is strongly indefinite, we can transform binary indefinite functional into unary functional by the methods of Benci and Fortunato in [5, 6]. In [1], Alves, Souto and Soares studied a similar system and found the Ambrosetti Rabinowitz condition is not a necessary condition for the existence of the ground state solution in this system. In [12], Jiang and Liu used the truncating tool and the variant form of Clark's theorem to obtain the existence of a series of nontrivial solutions that converge to 0 for the Schrödinger-Poisson system. In [21], Ye and Tang studied the existence and multiplicity of solutions for a system in $\mathbb{R}^3$ which is different from the system (1) and has parameter $\lambda > 0$ and $K(x)$, where nonlinearity $g(x,t)$ can be superlinear or sublinear, as $|t| \to \infty$. In [22], Zhao and Zhao studied the existence of the positive solution for the Schrödinger-Poisson system with a critical Sobolev exponent. In [13], Lin and Tang studied the existence of the ground state solution for the system (1) with $k(x) \equiv 0$. In [20], Xie, Chen and Shi studied the existence and multiplicity of the normalized solutions for the following system

$$\begin{cases} -\Delta t - \lambda t + K\phi t = g(x,t), & \text{in } \mathbb{R}^3, \\ -\Delta \phi_t = 4\pi t^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $K > 0$, g is superlinear and satisfies the monotonicity condition. Chen, Wu, and Jhang used the Linking Theorem in [8] to prove the existence of both the infinitely many high energy solutions and the nontrivial solutions for the Schrödinger equation.

Based on the research of many articles mentioned above, it can be seen that many scholars have made significant contributions to the study of these equations. And motivated by [9], we will consider the existence of the nontrivial solutions for the system (1).

Assuming the potential V satisfies the following conditions:

(A_1) $V \in C(\mathbb{R}^N)$, and $V(x) < V_\infty$, where $V_\infty := \lim\limits_{|x|\to\infty} V(x) < \infty$. Moreover, $0 \notin \sigma(-\Delta + V)$, where σ denotes the spectrum of the Schrödinger operator $-\Delta + V$.

Remark 1.1. *By (A_1), we can see 0 is not a spectrum for the $-\Delta + V$, this indicates that there is no zero space. It can be noted that the eigenvalue issue*

$$-\Delta t + V(x)t = \lambda t,$$

for $t \in U$, there is a complete eigenvalue sequence

$$-\infty < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l < 0 < \lambda_{l+1} \leq \cdots < \lambda_\infty.$$

e_j is defined as corresponding eigenfunction of λ_j and $|e_j|_2 = 1$.

Here we define the nondecreasing minimax sequence

$$\lambda_n = \lim_{T\in T_n} \sup_{t\in T\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla t|^2 + V(x)t^2\right) \mathrm{d}x}{\int_{\mathbb{R}^N} t^2 \mathrm{d}x},$$

where T_n represents a family of n-dimensional subspaces of $C_0^\infty(\mathbb{R}^N)$. By (A_1), it is evident that $\sigma_{ess}(-\Delta + V) \in (V_\infty, \infty)$. For any $\lambda_n < \lambda_\infty$, $\lambda_\infty := \lim\limits_{n\to\infty} \lambda_n = \inf \sigma_{ess}(-\Delta + V)$, and $\lambda_n \in \sigma_{pp}(-\Delta + V)$, where $\sigma_{ess}(-\Delta + V)$ is defined as the essential spectrum for $-\Delta + V$, and $\sigma_{pp}(-\Delta + V)$ is defined as the pure point spectrum for $-\Delta + V$(for more details, see [16, 17]).

Denote $G(x,z) := \int_0^z g(x,y)\mathrm{d}y$, and $2^* := \frac{2N}{N-2}$. We make the following assumptions about the nonlinearity g:

(A_2) $g(x,z) \in C^1(\mathbb{R}^N \times \mathbb{R})$, there is a constant $C_1 \geq 0$ and $\forall\, (x,z) \in \mathbb{R}^N \times \mathbb{R}$, such that

$$|g(x,z)| \leq C_1(1+|z|^{p-1}),$$

and as $z \to 0$, satisfy $g(x,z) = o(|z|)$, where $2 < p < 2^*$.

(A_3) For any $x \in \mathbb{R}^N$, $\lim\limits_{|z|\to\infty} \dfrac{G(x,z)}{z^4} = +\infty$.

(A_4) $\lim\limits_{|x|\to\infty} \sup\limits_{|z|\leq l} \dfrac{|g(x,z)|}{|z|} = 0$ for any $l > 0$.

(A_5) For $x \in \mathbb{R}^N$ and $z \neq 0$, there exist $a, b > 0$ and $\alpha \in (0, \beta)$ such that

$$0 < \left(4 + \frac{1}{a\,|z|^{\alpha} + b}\right) G(x,z) \leq z g(x,z),$$

where $\beta = \min\{p', 2^*(p'-1) - p'\}$, and $p' := \dfrac{p}{p-1}$ is the conjugate exponent for p.

In this paper, we always denote $U := H^1(\mathbb{R}^N)$ and define the functional $\Phi : U \to \mathbb{R}$ is as follows

$$\Phi(t) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla t|^2 + V(x)t^2\right) \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N} \phi_t t^2 \mathrm{d}x - \int_{\mathbb{R}^N} G(x,t)\mathrm{d}x - \int_{\mathbb{R}^N} k(x) t \mathrm{d}x, \tag{2}$$

where ϕ_t is the unique solution of the Poisson equation.

$$-\Delta\phi_t = t^2, \quad \forall\, t \in U,$$

by the Evans [11] Theorem 2.2.1,

$$\phi_t(x) = \int_{\mathbb{R}^N} \frac{t^2(y)}{4\pi|x-y|}\mathrm{d}y.$$

In this paper, one major difficulty is that the sign of V is indefinite, which causes serious challenges to study the solution of the system (1). Therefore, we adopt the idea of space decomposition to discuss the functional corresponding to the system (1) in different spaces.

Let $U^- = \operatorname{span}\{e_1, \cdots, e_l\}$, $U^+ = (U^-)^{\perp}$. So U^- and U^+ are the negative and positive spaces of the quadratic form Q on U, where

$$Q(t) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla t|^2 + V(x)t^2\right) \mathrm{d}x.$$

Notice that $\dim U^- < \infty$. In addition, there exists a constant $\kappa > 0$ such that

$$\pm Q(t) \geq \kappa \|t\|^2, \quad t \in U^{\pm}. \tag{3}$$

For $s, t \in U$ define the inner product

$$(s,t) = \int_{\mathbb{R}^N} \left(\nabla s^+ \nabla t^+ + V(x) s^+ t^+\right) \mathrm{d}x - \int_{\mathbb{R}^N} \left(\nabla s^- \nabla t^- + V(x) s^- t^-\right) \mathrm{d}x,$$

where $t = t^+ + t^-$, $t^- \in U^-$ and $t^+ \in U^+$. Thus U can be considered a Hilbert space with the norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$, and

$$\Phi(t) = \frac{1}{2}\left\|t^+\right\|^2 - \frac{1}{2}\left\|t^-\right\|^2 + \frac{1}{4}\int_{\mathbb{R}^N} \phi_t t^2 \mathrm{d}x - \int_{\mathbb{R}^N} G(x,t)\mathrm{d}x - \int_{\mathbb{R}^N} k(x) t \mathrm{d}x.$$

The imbedding $U \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for each $p \in [2, 2^*]$, moreover $\gamma_p > 0$, we have

$$|t|_p \leq \gamma_p \|t\|, \qquad \forall\, t \in U, \tag{4}$$

where L^p norm is represented by $|\cdot|_p$. Next, our conclusion is as follows

Theorem 1.1. *Assume that $k(x) \in L^2(\mathbb{R}^N)$, $k(x) \not\equiv 0$ and conditions $(A_1) - (A_5)$ are satisfied. Then there exists a constant $k_0 > 0$ such that the system (1) has at least one nontrivial solution in U whenever $|k|_2 < k_0$.*

2 Preliminaries

In this part, we will review a few basic facts in the theory of critical points, which will be used to verify our main conclusion.

Proposition 2.1. *([7, Lemma 2.2]) There is a constant $C_0 > 0$ such that*

$$0 \leq \frac{1}{4}\int_{\mathbb{R}^N} \phi_t t^2 \mathrm{d}x \leq C_0 \|t\|^4, \qquad \forall\, t \in U. \tag{5}$$

Lemma 2.1. *([19, Lemma 1]) Assume that $k(x) \in L^2(\mathbb{R}^N)$, (A_1) and (A_2) are satisfied. Hence, Φ is properly defined on U and $\Phi \in C^1(U, \mathbb{R})$, for each $s, t \in U$,*

$$\left(\Phi'(t), s\right) = \int_{\mathbb{R}^N} \left(\nabla t \nabla s + V(x) st\right) \mathrm{d}x + \int_{\mathbb{R}^N} \phi_t ts \mathrm{d}x - \int_{\mathbb{R}^N} g(x,t)s \mathrm{d}x - \int_{\mathbb{R}^N} k(x) s \mathrm{d}x. \tag{6}$$

Proof. On the basis of (2), set $\Psi(t) = \int_{\mathbb{R}^N} G(x,t)\mathrm{d}x + \int_{\mathbb{R}^N} k(x)t\mathrm{d}x$. To prove $\Phi \in C^1(U, \mathbb{R})$ and (6), we just to prove that for $\forall\, s, t\ \in U$,

$$\Psi \in C^1(U, \mathbb{R}), \qquad \left(\Psi'(t), s\right) = \int_{\mathbb{R}^N} \left(g(x,t) + k(x)\right) s \mathrm{d}x.$$

By (A_2), for $\varepsilon > 0$ and $\forall\, (x,t) \in \mathbb{R}^N \times \mathbb{R}$, there is a constant $A(\varepsilon)$ with ε,

$$|g(x,t)| \leq \varepsilon |t| + A(\varepsilon)|t|^{p-1}, \tag{7}$$

$$|G(x,t)| \leq \frac{\varepsilon}{2}|t|^2 + \frac{A(\varepsilon)}{p}|t|^p. \tag{8}$$

For any $s, t \in U$ and $0 < |h| < 1$, set $\eta = \dfrac{|(G(x,t+hs) + k(x)(t+hs)) - (G(x,t) + k(x)t)|}{|h|}$, according to (7) and the Lagrange mean value theorem, there exists $0 < \theta < 1$ such that

$$\begin{aligned}
\eta &\leq |g(x, t + \theta hs)s| + |k(x)s| \\
&\leq \left(\varepsilon\, |t + \theta hs| + A(\varepsilon)\, |t + \theta hs|^{p-1}\right) |s| + |k(x)s| \\
&\leq \varepsilon \left(|t|\, |s| + |s|^2\right) + 2^{p-1} A(\varepsilon) \left(|t|^{p-1}\, |s| + |s|^p\right) + |k(x)|\, |s|\,.
\end{aligned}$$

According to the Hölder inequality, we have

$$\eta \leq \varepsilon \left(|t|\, |s| + |s|^2\right) + 2^{p-1} A(\varepsilon) \left(|t|^{p-1}\, |s| + |s|^p\right) + |k(x)|\, |s| \in L^1(\mathbb{R}^N).$$

Therefore, by the Lebesgue Theorem, we have

$$\left(\Psi'(t), s\right) = \int_{\mathbb{R}^N} \left(g(x,t) + k(x)\right) s \mathrm{d}x.$$

Furthermore, $\Psi'(\cdot) : U \to U^*$ is continuous. Assume that $t_n \to t$ in U, since $U \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in [2, 2^*]$, we obtain

$$t_n \to t \quad \text{in} \quad L^p(\mathbb{R}^N).$$

Note that

$$\begin{aligned}\left\|\Psi'(t_n)-\Psi'(t)\right\| &= \sup_{\|s\|\le 1}\left|\int_{\mathbb{R}^N}(g(x,t_n)s+k(x)s)\,\mathrm{d}x-\int_{\mathbb{R}^N}(g(x,t)s+k(x)s)\,\mathrm{d}x\right| \\ &\le \sup_{\|s\|\le 1}\int_{\mathbb{R}^N}|g(x,t_n)-g(x,t)|\,|s|\,\mathrm{d}x.\end{aligned}$$

According to the Theorem A.4 in [18] and the Hölder inequality, we have

$$\sup_{\|s\|\le 1}\int_{\mathbb{R}^N}|g(x,t_n)-g(x,t)|\,|s|\,\mathrm{d}x\to 0, \qquad n\to\infty.$$

Hence $\|\Psi'(t_n)-\Psi'(t)\|\to 0$. The proof is completed. □

The critical point theory and the Local linking theorem are very important tools for solving elliptic partial differential equations. Here, we briefly explain the Local linking Theorem, which can be found in detail in the reference [10]. The commonly used (PS) condition is also an important condition to prove the existence of solutions. Here, we introduce the weaker condition (C) than the (PS) condition.

For direct sum decomposition space $U=U^-\oplus U^+$, we call the functional Φ has a local linking at 0, if there exists a positive constant $\rho>0$ such that

$$\begin{cases}\Phi(t)\le 0, & \text{for } t\in U^-,\ \|t\|\le\rho,\\ \Phi(t)>0, & \text{for } t\in U^+,\ 0<\|t\|\le\rho.\end{cases} \tag{9}$$

For Φ, 0 is known to be a trivial critical point.

Definition 2.1 ([10])**.** *For a Banach space U, and $\Phi\in C^1(U,\mathbb{R})$. A sequence $\{t_n\}\subset U$ with $c\in\mathbb{R}$, we call the sequence $\{t_n\}$ is a $(C)_c$ sequence, if*

$$\Phi(t_n)\to c, \quad (1+\|t_n\|)\left\|\Phi'(t_n)\right\|\to 0. \tag{10}$$

If there is a convergent subsequence for each $(C)_c$ sequence of Φ, then the functional Φ is said to satisfy the condition $(C)_c$. We call functional Φ satisfies the condition (C), if the functional Φ satisfies condition $(C)_c$ for every $c\in\mathbb{R}$.

Theorem 2.1. *([15, Theorem 2.2]) Assume $\Phi\in C^1(U,\mathbb{R})$ and the functional Φ has a local linking at 0, and that it maps bounded sets into bounded sets, and the Φ satisfies the condition (C). For every $m\in\mathbb{N}$ and $t\in U^-\oplus U_m^+$, we have*

$$\Phi(t)\to-\infty, \qquad \|t\|\to\infty. \tag{11}$$

Then Φ has a nontrivial critical point.

3 Proof of main result

In this section, we use the conditions that are obtained from the previous two sections to prove our main conclusion.

Lemma 3.1. *Assume that $k(x)\in L^2(\mathbb{R}^N)$ and (A_2), (A_3) are satisfied. Based on decomposition $U=U^-\oplus U^+$, there exist constants ρ and $\delta>0$ such that the functional $\Phi(t)$ has a local linking at 0, whenever $\|t\|\le\rho$ and $|k|_2<\delta$.*

Proof. Given that $k(x) \in L^2(\mathbb{R}^N)$ and $k(x) \not\equiv 0$, $t \in U$ can be chosen such that

$$\int_{\mathbb{R}^N} k(x)t(x)\mathrm{d}x > 0.$$

For $\varepsilon := \frac{\kappa}{\gamma_2^2} > 0$, if $t \in U^-$, using (3), (5), (8) and Hölder inequality, for $\|t\| \leq \rho$ (where $\rho > 0$) small enough, we have

$$\begin{aligned}
\Phi(t) &= Q(t) + \frac{1}{4}\int_{\mathbb{R}^N} \phi_t t^2\mathrm{d}x - \int_{\mathbb{R}^N} G(x,t)\mathrm{d}x - \int_{\mathbb{R}^N} k(x)t\mathrm{d}x \\
&\leq -\kappa \|t\|^2 + C_0 \|t\|^4 + \frac{\varepsilon}{2}|t|_2^2 + \frac{A(\varepsilon)}{p}|t|_p^p \\
&\leq -\frac{\kappa}{2}\|t\|^2 + C_0\|t\|^4 + C\|t\|^p,
\end{aligned}$$

where $C = \frac{A(\varepsilon)\gamma_p^p}{p}$. By (A_3), we can deduce $p > 4$, then we have $\Phi(t) \leq 0$. Similarly, for $t \in U^+$,

$$\begin{aligned}
\Phi(t) &= Q(t) + \frac{1}{4}\int_{\mathbb{R}^N} \phi_t t^2\mathrm{d}x - \int_{\mathbb{R}^N} G(x,t)\mathrm{d}x - \int_{\mathbb{R}^N} k(x)t\mathrm{d}x \\
&\geq \kappa\|t\|^2 - \frac{\varepsilon}{2}|t|_2^2 - \frac{A(\varepsilon)}{p}|t|_p^p - |k|_2|t|_2 \\
&\geq \frac{\kappa}{2}\|t\|^2 - C\|t\|^p - \gamma_2|k|_2\|t\| \\
&= \|t\|\left(\frac{\kappa}{2}\|t\| - C\|t\|^{p-1} - \gamma_2|k|_2\right).
\end{aligned}$$

Take

$$h(t) = \frac{\kappa}{2}t - Ct^{p-1}, \forall\, t \geq 0.$$

Observing that $4 < p < 2^*$, we deduce that

$$h(\|t\|) = \frac{\kappa}{2}\|t\| - C\|t\|^{p-1} > 0.$$

Therefore, take $\delta := \frac{1}{2\gamma_2}h(\|t\|) > 0$, it has

$$\Phi(t) \geq \frac{1}{2}\|t\|\,h(\|t\|) > 0,$$

whenever $|k|_2 < \delta$. This completes the proof. □

Lemma 3.2. *Assume that $(A_1) - (A_5)$ are satisfied, then any $(C)_c$ sequence of the functional Φ with $c \in \mathbb{R}$ is bounded.*

Proof. Using (A_5), for $t_n \neq 0$ and $x \in \mathbb{R}^N$,

$$\left(\frac{4a|t_n|^\alpha + 4b}{4a|t_n|^\alpha + 4b + 1}\right)t_n g(x,t_n) \geq G(x,t_n).$$

After a simple calculation, we have

$$t_n g(x,t_n) - 4G(x,t_n) \geq \frac{1}{4a|t_n|^\alpha + 4b + 1}t_n g(x,t_n) > 0. \tag{12}$$

Consider a $(C)_c$ sequence of the functional Φ as $\{t_n\} \subset U$, for every $c \in \mathbb{R}$,

$$\Phi(t_n) \to c, \quad (1 + \|t_n\|) \left\|\Phi'(t_n)\right\| \to 0.$$

To prove the conclusion we may assume $\|t_n\| \to \infty$ for a contradiction. Let $s_n = \dfrac{t_n}{\|t_n\|}$, up to a subsequence, we set $\{s_n\} \subset U$, $s \in U$, and $s_n \rightharpoonup s$, where

$$s_n = s_n^+ + s_n^-, \quad s_n^\pm \in U^\pm.$$

Similarly, $s = s^+ + s^-$, $s^\pm \in U^\pm$. For $\dim U^- < \infty$, note that if $s = 0$, we have $s_n^- \to s^- = 0$. Since

$$\|s_n\|^2 = \|s_n^-\|^2 + \|s_n^+\|^2 = 1,$$

for n is sufficiently large, we obtain

$$\|s_n^+\|^2 - \|s_n^-\|^2 \geq \frac{1}{2}.$$

Using (12), for n large enough, we deduce

$$\begin{aligned}
4c + 1 + \|t_n\| &\geq 4\Phi(t_n) - \big(\Phi'(t_n), t_n\big) \\
&= \int_{\mathbb{R}^N} \left(|\nabla t_n|^2 + V(x)t_n^2\right) \mathrm{d}x + \int_{\mathbb{R}^N} \left(g(x,t_n)t_n - 4G(x,t_n)\right) \mathrm{d}x - 3\int_{\mathbb{R}^N} k(x)t_n \mathrm{d}x \\
&\geq \left\|t_n^+\right\|^2 - \left\|t_n^-\right\|^2 - 3\,|k|_2\,|t_n|_2 \\
&\geq \frac{1}{2}\,\|t_n\|^2 - 3\gamma_2\,|k|_2\,\|t_n\|\,.
\end{aligned}$$

Then

$$\frac{1}{2}\,\|t_n\|^2 \leq 4c + 1 + (1 + 3\gamma_2\,|k|_2)\,\|t_n\| \leq 4c + 1 + (1 + 3\gamma_2\delta)\,\|t_n\|\,, \tag{13}$$

it is known that $\{t_n\}$ is bounded, which contradiction to $\|t_n\| \to \infty$, thus $s \neq 0$.

Set $\Lambda = \{x \in \mathbb{R}^N : s(x) \neq 0\}$. Then for $x \in \Lambda$, we have $|t_n(x)| \to +\infty$ and by (A_3)

$$\int_{s\neq 0} \frac{G(x,t_n(x))}{t_n^4(x)} s_n^4(x)\mathrm{d}x \to +\infty. \tag{14}$$

However, for n large enough, we have

$$\begin{aligned}
\int_{s\neq 0} \frac{G(x,t_n(x))}{t_n^4(x)} s_n^4(x)\mathrm{d}x &= \frac{1}{\|t_n\|^4}\int_{s\neq 0} G(x,t_n(x))\mathrm{d}x \leq \frac{1}{\|t_n\|^4}\int_{\mathbb{R}^N} G(x,t_n(x))\mathrm{d}x \\
&= \frac{1}{\|t_n\|^4}\left(\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla t_n|^2 + V(x)t_n^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N}\phi_{t_n}t_n^2\mathrm{d}x - \int_{\mathbb{R}^N} k(x)t_n\mathrm{d}x - \Phi(t_n)\right) \\
&\leq \frac{1}{2\,\|t_n\|^2} + \frac{1}{4\,\|t_n\|^4}\int_{\mathbb{R}^N}\phi_{t_n}t_n^2\mathrm{d}x + \frac{\gamma_2\,|k|_2\,\|t_n\|}{\|t_n\|^4} \\
&\leq 2 + C_0,
\end{aligned}$$

which contradiction to (14), then $\{t_n\}$ is bounded. □

Lemma 3.3. *Assume that the hypotheses $(A_1)-(A_5)$ are satisfied, then the functional Φ satisfies the condition (C).*

Proof. By the Lemma 3.2, set $\{t_n\}$ is a $(C)_c$ sequence of Φ that is bounded. Up to a subsequence, we can assume $t_n \rightharpoonup t$ in U. Due to $\dim U^- < \infty$, then

$$t_n^- \to t^- \text{ in } U^-; \quad t_n^+ \rightharpoonup t^+ \text{ in } U^+; \quad t_n^+ \to t^+ \text{ in } L^p_{loc}(\mathbb{R}^N), \quad p \in [2, 2^*).$$

Next, we just need to prove that

$$\|t_n^+\| \to \|t^+\|. \tag{15}$$

Since $t_n^+ \rightharpoonup t^+$, then

$$\int_{\mathbb{R}^N} \left(\nabla t_n^+ \nabla t^+ + V(x) t_n^+ t^+\right) \mathrm{d}x \to \int_{\mathbb{R}^N} \left(\left|\nabla t^+\right|^2 + V(x)\left|t^+\right|^2\right) \mathrm{d}x = \left\|t^+\right\|^2.$$

Form (6), it has

$$\begin{aligned}
o(1) &= \left(\Phi'(t_n) - \Phi'(t), t_n^+ - t^+\right) \\
&= \int_{\mathbb{R}^N} \left(\nabla t_n \nabla \left(t_n^+ - t^+\right) + V(x) t_n \left(t_n^+ - t^+\right)\right) \mathrm{d}x + \int_{\mathbb{R}^N} \phi_{t_n} t_n \left(t_n^+ - t^+\right) \mathrm{d}x - \int_{\mathbb{R}^N} g(x, t_n)\left(t_n^+ - t^+\right) \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \left(\nabla t \nabla \left(t_n^+ - t^+\right) + V(x) t \left(t_n^+ - t^+\right)\right) \mathrm{d}x - \int_{\mathbb{R}^N} \phi_t t \left(t_n^+ - t^+\right) \mathrm{d}x + \int_{\mathbb{R}^N} g(x, t)\left(t_n^+ - t^+\right) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left(\left|\nabla t_n^+\right|^2 + V(x)\left|t_n^+\right|^2\right) \mathrm{d}x + \int_{\mathbb{R}^N} \left(\left|\nabla t^+\right|^2 + V(x)\left|t^+\right|^2\right) \mathrm{d}x - 2\int_{\mathbb{R}^N} \left(\nabla t_n^+ \nabla t^+ + V(x) t_n^+ t^+\right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left(\phi_{t_n} t_n - \phi_t t\right)\left(t_n^+ - t^+\right) \mathrm{d}x - \int_{\mathbb{R}^N} \left(g(x, t_n) - g(x, t)\right)\left(t_n^+ - t^+\right) \mathrm{d}x \\
&= \left\|t_n^+\right\|^2 - \left\|t^+\right\|^2 + \int_{\mathbb{R}^N} \left(\phi_{t_n} t_n - \phi_t t\right)\left(t_n^+ - t^+\right) \mathrm{d}x - \int_{\mathbb{R}^N} \left(g(x, t_n) - g(x, t)\right)\left(t_n^+ - t^+\right) \mathrm{d}x + o(1).
\end{aligned}$$

Collecting all infinitesimal terms, we obtain

$$\begin{aligned}
\left\|t_n^+\right\|^2 - \left\|t^+\right\|^2 &= \int_{\mathbb{R}^N} \left(g(x, t_n) - g(x, t)\right)\left(t_n^+ - t^+\right) \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \left(\phi_{t_n} t_n - \phi_t t\right)\left(t_n^+ - t^+\right) \mathrm{d}x + o(1).
\end{aligned} \tag{16}$$

Now, we let $\varepsilon > 0$, for $l \geq 1$, by (A_2) and Hölder inequality, we have

$$\begin{aligned}
\int_{|t_n| \geq l} g(x, t_n)\left(t_n^+ - t^+\right) \mathrm{d}x &\leq 2C_1 \int_{|t_n| \geq l} |t_n|^{p-1}\left|t_n^+ - t^+\right| \mathrm{d}x \\
&\leq 2C_1 l^{p-2^*} \int_{|t_n| \geq l} |t_n|^{2^*-1}\left|t_n^+ - t^+\right| \mathrm{d}x \\
&\leq 2C_1 l^{p-2^*} |t_n|_{2^*}^{2^*-1}\left|t_n^+ - t^+\right|_{2^*}.
\end{aligned}$$

Since $p < 2^*$, we can set l to a large enough value to ensure

$$\int_{|t_n| \geq l} g(x, t_n)\left(t_n^+ - t^+\right) \mathrm{d}x \leq \frac{\varepsilon}{3}, \tag{17}$$

for arbitrary n. Additionally, by (A_4) there is a constant $L > 0$ such that

$$\int_{\substack{|x| \geq L \\ |t_n| \leq l}} g(x, t_n)\left(t_n^+ - t^+\right) \mathrm{d}x \leq |t_n|_2 \left|t_n^+ - t^+\right|_2 \sup_{|z| \leq l, |x| \geq L} \frac{|g(x, z)|}{|z|} \leq \frac{\varepsilon}{3}, \tag{18}$$

for arbitrary n. Finally, for $p \in [2, 2^*)$, we have $t_n^+ \to t^+$ in $L^p(B_L(0))$, by (7) it follows that

$$\begin{aligned}\int_{\substack{|x|\leq L\\|t_n|\leq l}} g(x,t_n)\left(t_n^+ - t^+\right) \mathrm{d}x &\leq \int_{\substack{|x|\leq L\\|t_n|\leq l}} \left(\varepsilon\,|t_n|\left|t_n^+ - t^+\right| + A(\varepsilon)\,|t_n|^{p-1}\left|t_n^+ - t^+\right|\right) \mathrm{d}x\\ &\leq \varepsilon\,|t_n|_2\left|t_n^+ - t^+\right|_{L^2(B_L(0))} + A(\varepsilon)\,|t_n|_p^{p-1}\left|t_n^+ - t^+\right|_{L^p(B_L(0))}\\ &\leq \frac{\varepsilon}{3},\end{aligned} \tag{19}$$

for n large enough. Putting (17)-(19) together, we notice that

$$\int_{\mathbb{R}^N} g(x,t_n)\left(t_n^+ - t^+\right) \mathrm{d}x \leq \varepsilon.$$

Similarly, $\int_{\mathbb{R}^N} g(x,t)\left(t_n^+ - t^+\right) \mathrm{d}x \leq \varepsilon$. Therefore, for n large enough, we have

$$\int_{\mathbb{R}^N} \left(g(x,t_n) - g(x,t)\right)\left(t_n^+ - t^+\right) \mathrm{d}x \leq \varepsilon.$$

Similar to the method of [1], since $t_n \rightharpoonup t$ in U, then $t_n \to t$ in $L^p_{loc}(\mathbb{R}^N)$, for $p \in [2, 2^*)$. Taking $v \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla\left(\phi_{t_n} - \phi_t\right)\nabla v \mathrm{d}x &= \int_{\mathbb{R}^N} \left(t_n^2 - t^2\right) v \mathrm{d}x\\ &\leq \left(\int_{supp\ v} \left(t_n - t\right)^2 \mathrm{d}x\right)^{1/2}\left(\int_{supp\ v} \left(t_n + t\right)^2 \mathrm{d}x\right)^{1/2} |v|_\infty\\ &\to 0, \qquad \text{as}\ \ n \to \infty.\end{aligned}$$

Thus $\phi_{t_n} \rightharpoonup \phi_t$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, as $n \to \infty$, then we have

$$\int_{\mathbb{R}^N} \phi_{t_n} t v \mathrm{d}x \to \int_{\mathbb{R}^N} \phi_t t v \mathrm{d}x.$$

Moreover, by Hölder inequality we get

$$\int_{\mathbb{R}^N} \phi_{t_n}\left(t_n - t\right) v \mathrm{d}x \leq |\phi_{t_n}|_{2^*}\,|t_n - t|_{L^{\frac{2\cdot 2^*}{2^*-1}}(\Omega)}\,|v|_{L^{\frac{2\cdot 2^*}{2^*-1}}(\Omega)} \to 0,$$

where Ω denotes the compact support of the v. Therefore

$$\int_{\mathbb{R}^N} \left(\phi_{t_n} t_n v - \phi_t t v\right) \mathrm{d}x = \int_{\mathbb{R}^N} \left(\phi_{t_n} - \phi_t\right) t v \mathrm{d}x + \int_{\mathbb{R}^N} \phi_{t_n}\left(t_n - t\right) v \mathrm{d}x \to 0.$$

So, we have

$$\int_{\mathbb{R}^N} \left(\phi_{t_n} t_n - \phi_t t\right)\left(t_n^+ - t^+\right) \mathrm{d}x \leq \varepsilon.$$

From (16) we deduce that (15). □

Lemma 3.4. *Assume X be a finite dimensional subspace of U, then we have*

$$\Phi(t) \to -\infty, \qquad \textit{as}\ \ \|t\| \to \infty, \quad t \in X.$$

Proof. If the conclusion is not true, we can select $\{t_n\} \subset X$ and $\mu \in \mathbb{R}$ such that

$$\|t_n\| \to \infty, \quad \Phi(t_n) \geq \mu. \tag{20}$$

Let $s_n = \dfrac{t_n}{\|t_n\|}$. Since $\dim X < +\infty$, up to a subsequence, there is $s \in X$ with $\|s\| = 1$ such that

$$s_n(x) \to s(x) \quad \text{in} \quad X, \quad s_n(x) \to s(x) \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^N.$$

Since $s \neq 0$, from (14),

$$\int_{\mathbb{R}^N} \frac{G(x, t_n(x))}{\|t_n\|^4} \mathrm{d}x \to +\infty. \tag{21}$$

By (5) and (21), we deduce

$$\begin{aligned}
\Phi(t_n) &= \|t_n\|^4 \left(\frac{\|t_n^+\|^2 - \|t_n^-\|^2}{2\|t_n\|^4} + \frac{1}{4\|t_n\|^4} \int_{\mathbb{R}^N} \phi_{t_n} t_n^2 \mathrm{d}x - \int_{\mathbb{R}^N} \frac{G(x, t_n)}{\|t_n\|^4} \mathrm{d}x - \int_{\mathbb{R}^N} \frac{k(x) t_n}{\|t_n\|^4} \mathrm{d}x \right) \\
&\leq \|t_n\|^4 \left(\frac{1}{2\|t_n\|^2} + C_0 + \frac{\gamma_2 |k|_2}{\|t_n\|^3} - \int_{\mathbb{R}^N} \frac{G(x, t_n)}{\|t_n\|^4} \mathrm{d}x \right) \to -\infty,
\end{aligned}$$

which contradiction to (20), this completes the proof. □

Theorem 1.1 can be proved.

Proof. We can via the Theorem 2.1 to gain the conclusion of the Theorem 1.1. It is evident from the Proposition 2.1 that the functional Φ maps bounded sets into bounded sets. From the Lemma 3.1, the functional Φ has a Local linking at 0 whenever $\|t\| \leq \rho$ and $|k|_2 < \delta = k_0$. And from the Lemma 3.3, we can see the functional Φ satisfies the condition (C). Because $\dim(U^- \oplus U_m^+) < \infty$, then (11) is true via the Lemma 3.4. It suffices to verify the Theorem 1.1 by the Theorem 2.1. □

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