Research article

ISSN: 2980-1982
2024, Vol. 1, No. 2, 21–34

Letters on
Applied and Pure Mathematics

Parameter-uniform convergence analysis on a Bakhvalov-type mesh with a smooth mesh-generating function using the preconditioning approach

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Abstract

The linear singularly perturbed convection-diffusion problem in one dimension is considered and its discretization on a Bakhvalov-type mesh generated by a smooth mesh-generating function is analyzed. The preconditioning technique is used to obtain the first-order pointwise convergence uniform in the perturbation parameter.

Key words: singular perturbation, convection-diffusion, boundary-value problem, Bakhvalov-type mesh, finite differences, uniform convergence, preconditioning
2020 Mathematics Subject Classification: 65L10, 65L12, 65L20, 65L70
Article history: Received 10 Jan 2024; Accepted 27 Feb 2024; Online 25 March 2024

1 Introduction

The goal of this paper is to provide a novel theoretical analysis based on a preconditioning approach developed by Vulanović and Nhan [17] for the one-dimensional singularly perturbed convection-diffusion problem,

$$\mathcal{L}u := -\varepsilon u'' - b(x)u' + c(x)u = f(x),\ x \in (0,1),\ u(0) = u(1) = 0, \tag{1.1}$$

where ε is a positive perturbation parameter, $0 < \varepsilon \leq 1$. We assume that the functions b, c, and f are sufficiently smooth, and that

$$b(x) \geq \beta > 0, \ \ c(x) \geq 0 \ \text{ for } x \in I := [0,1].$$

Under these assumptions, the boundary value problem (1.1) has a unique solution, $u \in C^2(I)$.

When ε is small, the problem (1.1) is convection-dominated and the solution u typically has an exponential boundary layer near $x = 0$. In the layer, the kth derivative of the solution behaves like $\mathcal{O}(\varepsilon^{-k})$. Because of this, singular perturbation problems require special numerical methods that are *parameter-robust*. Their goal is to achieve ε-uniform accuracy (ε-uniform convergence). The use of layer-adapted meshes, either Bakhvalov-type meshes or Shishkin-type meshes, is one of the most frequently used approaches to achieve the goal. The class of Bakhvalov-type meshes shares the same elegant property of its original mesh introduced by Bakhvalov [1] in 1969. That is, the Bakhvalov-type meshes are created by smooth mesh-generating functions. This feature distinguishes them from the Shishkin-type meshes which can be generated by piecewise differentiable functions. Because of this, Shishkin-type meshes are often simpler, but as a trade-off, their convergence rate is usually sub-optimal when compared to that of the Bakhvalov-type meshes.

Even when Bakhvalov-type or Shishkin-type meshes are used, the error analysis of finite-difference methods for the problem (1.1) is still challenging. The main problem is that, in contrast to the reaction-diffusion problems, ε-uniform convergence cannot be proved using the classical principle "ε-uniform stability and ε-uniform consistency imply ε-uniform convergence." This is because ε-uniform consistency is absent in the case of convection-diffusion problems of type (1.1). For instance, when (1.1) is discretized by the upwind scheme on the Shishkin mesh, the consistency (truncation) errors behave like $\mathcal{O}(\varepsilon^{-1}N^{-1}\ln N)$ where N is the discretization parameter (cf. [10, 17] for numerical observations of this phenomenon). This is why special techniques are devised to prove ε-uniform convergence for finite-difference schemes discretizing (1.1) on layer-adapted meshes. These include the hybrid-stability approach [5] and truncation-error and barrier functions [12, 13]. A relatively recent method of proof, introduced by Vulanović and Nhan [17], is to use the preconditioning technique. This idea is extended further to hybrid higher-order finite-difference schemes [10, 18] and to a method that uses a very special decomposition of the solution [19]. Note that as mentioned in our papers [18, 19], the preconditioning technique is the only analysis that works for more complicated schemes.

The meshes used in the above papers are the Shishkin mesh and its modifications. The result closest to a preconditioning-based proof for a Bakhvalov-type mesh is presented in [14]. The mesh considered there is the Bakhvalov-Shishkin mesh [4] that uses the explicit Shishkin transition point between the fine part of the mesh in the layer and the coarse part outsied the layer. This mesh is not generated by a smooth function like the original Bakhvalov mesh. By contrast, the goal of this article is to show that it is possible to generalize the preconditioning-driven analysis to a Bakhvalov-type mesh defined by a smooth mesh-generating function. The mesh we consider is the simplest one in the Bakhvalov mesh generalization by Vulanović [15]; we shall call it the Vulanović-Bakhvalov mesh. Because the preconditioning technique has proven its capability to handle more sophisticated schemes, our result might be employed to analyze more complicated higher-order methods, similarly to Vulanović and Nhan [18, 19] but on Bakhvalov-type meshes.

In the next section, we introduce the solution decomposition, the discrete problem, as well as the condition number estimates for an un-preconditioned discrete system. We then describe the Vulanović-Bakhvalov mesh in Section 3. A carefully selected preconditioner to scale the discrete system is presented in Section 4 and used to obtain ε-uniform stability. Finally, the preconditioned consistency error is analyzed and the uniform convergence result is derived in Section 5, where we also present results of a numerical test.

2 The solution decomposition, the discrete problem, and the condition number estimate

The solution u can be decomposed into the smooth and boundary-layer parts. We present here Linß's [4, Theorem 3.48] version of such a decomposition:

$$u(x) = s(x) + y(x), \tag{2.1}$$

$$|s^{(k)}(x)| \le C\left(1+\varepsilon^{2-k}\right), \quad |y^{(k)}(x)| \le C\varepsilon^{-k}e^{-\beta x/\varepsilon}, \tag{2.2}$$
$$x \in I, \quad k = 0,1,2,3.$$

Above and throughout the paper, C denotes a generic positive constant which is independent of ε. For the construction of the function s, see [4], since the details are not of interest here. As for y, it is important to note that it solves the problem

$$\mathcal{L}y(x) = 0, \quad x \in (0,1), \quad y(0) = -s(0), \quad y(1) = 0,$$

with a homogeneous differential equation. We shall use this fact later on in this paper.

We first define a finite-difference discretization of the problem (1.1) on a general mesh I^N with mesh points x_i, $i = 0,1,\dots,N$, such that $0 = x_0 < x_1 < \cdots < x_N = 1$. Throughout the rest of the paper, the constants C are also independent of N.

Let $h_i = x_i - x_{i-1}$, $i = 1,2,\dots,N$, and $\hbar_i = (h_i + h_{i+1})/2$, $i = 1,2,\dots,N-1$. Mesh functions on I^N are denoted by W^N, U^N, etc. If g is a function defined on I, we write g_i instead of $g(x_i)$ and g^N for the corresponding mesh function. Any mesh function W^N is identified with an $(N+1)$-dimensional column vector, $W^N = [W_0^N, W_1^N, \dots, W_N^N]^T$, and its maximum norm is given by

$$\left\|W^N\right\| = \max_{0\le i\le N} |W_i^N|.$$

For the matrix norm, which we also denote by $\|\cdot\|$, we take the norm subordinate to the above maximum vector norm.

We discretize the problem (1.1) on I^N using the upwind finite-difference scheme:

$$U_0^N = 0,$$
$$\mathcal{L}^N U_i^N := -\varepsilon D'' U_i^N - b_i D' U_i^N + c_i U_i^N = f_i, \quad i = 1,2,\dots,N-1, \tag{2.3}$$
$$U_N^N = 0,$$

where

$$D'' W_i^N = \frac{1}{\hbar_i}\left(\frac{W_{i+1}^N - W_i^N}{h_{i+1}} - \frac{W_i^N - W_{i-1}^N}{h_i}\right)$$

and

$$D' W_i^N = \frac{W_{i+1}^N - W_i^N}{h_{i+1}}.$$

The linear system (2.3) can be written down in matrix form,

$$A_N U^N = \hat{f}^N, \tag{2.4}$$

where $A_N = [a_{ij}]$ is a tridiagonal matrix with $a_{00} = 1$ and $a_{NN} = 1$ being the only nonzero elements in the 0th and Nth rows, respectively, and where $\hat{f}^N = [0, f_1, f_2, \ldots, f_{N-1}, 0]^T$.

It is easy to see that A_N is an L-matrix, i.e., $a_{ii} > 0$ and $a_{ij} \leq 0$ if $i \neq j$, for all $i, j = 0, 1, \ldots, N$. The matrix A_N is also inverse monotone, which means that it is non-singular and that $A_N^{-1} \geq 0$ (inequalities involving matrices and vectors should be understood component-wise), and therefore an M-matrix (inverse monotone L-matrix). This can be proved using the following M-criterion, see [2] for instance.

Theorem 2.1. *Let A be an L-matrix and let there exist a vector w such that $w > 0$ and $Aw \geq \gamma$ for some positive constant γ. A is then an M-matrix and it holds that $\|A^{-1}\| \leq \gamma^{-1}\|w\|$.*

To see that A_N is an M-matrix, just set $w_i = 2 - x_i$, $i = 0, 1, \ldots, N$ in Theorem 2.1 to get that $A_N w \geq \min\{1, \beta\}$. This also implies that the discrete problem (2.4) is stable uniformly in ε,

$$\|A_N^{-1}\| \leq \frac{2}{\min\{1, \beta\}} \leq C. \tag{2.5}$$

Of course, the system (2.4) has a unique solution U^N.

3 A Bakhvalov-type mesh

A generalization of the Bakhvalov mesh [1] to a class of Bakhvalov-type meshes can be found in [15]. Here we take one of the Bakhvalov-type meshes from [15] for the discretization mesh I^N. We refer to this mesh as Vulanović-Bakhvalov mesh (VB-mesh). The points of the VB-mesh are generated by the function λ in the sense that $x_i = \lambda(t_i)$, where $t_i = i/N$. The mesh-generating function λ is defined as follows:

$$\lambda(t) = \begin{cases} \psi(t), & t \in [0, \alpha], \\ \psi(\alpha) + \psi'(\alpha)(t - \alpha), & t \in [\alpha, 1], \end{cases} \tag{3.1}$$

with $0 < q < 1$ and $\psi = a\varepsilon\phi$, where

$$\phi(t) = \frac{t}{q - t} = \frac{q}{q - t} - 1, \quad t \in [0, \alpha].$$

On the interval $[\alpha, 1]$, λ is the tangent line from the point $(1, 1)$ to ψ, touching ψ at $(\alpha, \psi(\alpha))$. The point α can be determined from the equation

$$\psi(\alpha) + \psi'(\alpha)(1 - \alpha) = 1.$$

Since $\phi'(t) = q/(q - t)^2$, the above equation reduces to a quadratic one,

$$a\varepsilon\alpha(q - \alpha) + a\varepsilon q(1 - \alpha) = (q - \alpha)^2,$$

which is easy to solve for α:

$$\alpha = \frac{q - \sqrt{a\varepsilon q(1 - q + a\varepsilon)}}{1 + a\varepsilon}.$$

We have to assume that $a\varepsilon < q$ (which is equivalent to $\psi'(0) < 1$) and then $\alpha > 0$. Note also that $\alpha < q$ and

$$q - \alpha = \zeta\sqrt{\varepsilon}, \quad \zeta \leq C, \quad \frac{1}{\zeta} \leq C. \tag{3.2}$$

Let J be the index such that $t_{J-1} < \alpha \le t_J$. Starting from the mesh point x_J, the mesh is uniform, with step size H. However, x_J behaves differently from the transition point of the Shishkin mesh because

$$x_J \ge \psi(\alpha) = \frac{a\alpha}{\zeta}\sqrt{\varepsilon}.$$

We note that the transition point $\psi(\alpha)$ is different also from the Bakhvalov-Shishkin of Vulanović-Shishkin meshes in the sense of [8].

We now give the estimate for the condition number of A_N when the discrete problem (2.3) is formed on the VB-mesh as described above. The condition number is

$$\kappa(A_N) := \|A_N^{-1}\|\|A_N\|.$$

We estimate the upper bound for $\|A_N\|$ by examining the entries of the matrix A_N directly,

$$\|A_N\| \le C\frac{N^2}{\varepsilon}.$$

Combining this with (2.5), we get the following result.

Theorem 3.1. *The condition number of A_N on the VB-mesh satisfies the following sharp bound:*

$$\kappa(A_N) \le C\frac{N^2}{\varepsilon}.$$

4 Conditioning

Let $M = \text{diag}\,(m_0, m_1, \dots, m_N)$ be a diagonal matrix with the entries

$$m_0 = 1,\ \ m_i = \frac{\hbar_i}{H},\ i = 1, 2, \dots, N-1,\ \text{ and }\ m_N = 1.$$

In other words,

$$m_0 = 1,\ \ m_i = \frac{\hbar_i}{H},\ i = 1, 2, \dots, J,\ \text{ and }\ m_i = 1, i = J+1, \dots, N. \tag{4.1}$$

When the system (2.4) is multiplied by M, this is equivalent to multiplying the equations 1, 2, …, J of the discrete problem (2.3) by $\hbar_i/H, i = 1, 2, \dots, J$. The modified system is

$$\tilde{A}_N U^N = M\tilde{f}^N, \tag{4.2}$$

where $\tilde{A}_N = MA_N$. Let the entries of $\tilde{A}_N$ be denoted by $\tilde{a}_{ij}$, the nonzero ones being

$$l_i := \tilde{a}_{i,i-1} = \begin{cases} -\dfrac{\varepsilon}{h_i H}, & 1 \le i \le J-1, \\[2ex] -\dfrac{\varepsilon}{h_J H}, & i = J, \\[2ex] -\dfrac{\varepsilon}{H^2}, & J+1 \le i \le N-1, \end{cases}$$

$$r_i := \tilde{a}_{i,i+1} = \begin{cases} -\dfrac{\varepsilon}{h_{i+1}H} - \dfrac{b_i\hbar_i}{h_{i+1}H}, & 1 \le i \le J-1, \\[2ex] -\dfrac{\varepsilon}{H^2} - \dfrac{b_i\hbar_i}{H^2}, & i = J, \\[2ex] -\dfrac{\varepsilon}{H^2} - \dfrac{b_i}{H}, & J+1 \le i \le N-1, \end{cases}$$

and

$$d_i := \tilde{a}_{ii} = \begin{cases} 1, & i = 0 \\ -l_i - r_i + \frac{\hbar_i}{H} c_i, & 1 \le i \le J, \\ -l_i - r_i + c_i, & J+1 \le i \le N-1, \\ 1, & i = N. \end{cases}$$

Unlike the Shishkin mesh, which is piecewise uniform, the VB-mesh is graded in the fine part. Because of this, it is more difficult to prove the uniform stability of the modified scheme. This is done in Lemma 4.2 below, but first we need some crucial estimates for the graded mesh defined by (3.1).

Lemma 4.1. *For the mesh-generating function given in* (3.1)*, the following estimates hold true:*

$$\frac{\varepsilon(h_{i+1} - h_i)}{h_i h_{i+1}} \le \frac{2}{a}, \qquad i = 1, 2, \ldots, J-2, \tag{4.3}$$

and

$$\frac{\varepsilon(H - h_J)}{h_J H} \le \frac{\zeta\sqrt{\varepsilon}}{aq}. \tag{4.4}$$

Proof. For $i \le J-2$, we have

$$h_i = x_i - x_{i-1} = a\varepsilon\left(\frac{q}{q - t_i} - \frac{q}{q - t_{i-1}}\right) = \frac{a\varepsilon q}{N(q - t_{i-1})(q - t_i)},$$
$$h_{i+1} = \frac{a\varepsilon q}{N(q - t_i)(q - t_{i+1})},$$

and

$$h_{i+1} - h_i = \frac{2a\varepsilon q}{N^2(q - t_{i-1})(q - t_i)(q - t_{i+1})}.$$

Then (4.3) follows because

$$\frac{\varepsilon(h_{i+1} - h_i)}{h_i h_{i+1}} = \frac{2(q - t_i)}{aq} = \frac{2}{a}\left(1 - \frac{t_i}{q}\right) \le \frac{2}{a}.$$

The proof of (4.4) is more complicated due to the presence of h_J. First, $h_J = \gamma_1 + \gamma_2$, where $\gamma_1 = x_\alpha - x_{J-1}$, $\gamma_2 = x_J - x_\alpha$, and $x_\alpha = \psi(\alpha)$. Since

$$\begin{aligned} \gamma_2 &= \psi'(\alpha)(t_J - \alpha) \\ &= \frac{a\varepsilon q}{q - \alpha}\left(\frac{t_J - \alpha}{q - \alpha}\right) \end{aligned}$$

and

$$\begin{aligned} \gamma_1 &= a\varepsilon\left(\phi(\alpha) - \phi\left(t_{J-1}\right)\right) \\ &= a\varepsilon\left(\frac{\alpha}{q - \alpha} - \frac{t_{J-1}}{q - t_{J-1}}\right) \\ &= \frac{a\varepsilon q}{q - \alpha} \cdot \frac{\alpha - t_{J-1}}{q - t_{J-1}}, \end{aligned}$$

we have

$$\begin{aligned} h_J &= \frac{a\varepsilon q}{q-\alpha}\left[\frac{t_J-\alpha}{q-\alpha}+\frac{\alpha-t_{J-1}}{q-t_{J-1}}\right] \\ &= \frac{a\varepsilon q}{(q-\alpha)^2}\left[t_J-\alpha+\frac{(q-\alpha)(\alpha-t_{J-1})}{q-t_{J-1}}\right] \\ &= \frac{a\varepsilon q}{\zeta^2}\left[t_J-\alpha+\frac{\zeta\sqrt{\varepsilon}(\alpha-t_{J-1})}{q-t_{J-1}}\right]. \end{aligned}$$

Moreover,

$$\psi'(\alpha)=\frac{a\varepsilon q}{(q-\alpha)^2} \quad \text{and } H=x_{J+1}-x_J=\frac{\psi'(\alpha)}{N},$$

implying that

$$H=\frac{a\varepsilon q}{N(q-\alpha)^2}.$$

Therefore,

$$\begin{aligned} H-h_J &= \frac{a\varepsilon q}{q-\alpha}\left[\frac{1}{N(q-\alpha)}-\frac{t_J-\alpha}{q-\alpha}-\frac{\alpha-t_{J-1}}{q-t_{J-1}}\right] \\ &= \frac{a\varepsilon q}{q-\alpha}\left[\frac{\alpha-t_{J-1}}{q-\alpha}-\frac{\alpha-t_{J-1}}{q-t_{J-1}}\right] \\ &= \frac{a\varepsilon q}{q-\alpha}\left(\alpha-t_{J-1}\right)\left[\frac{1}{q-\alpha}-\frac{1}{q-t_{J-1}}\right] \\ &= \frac{a\varepsilon q}{q-\alpha}\left(\alpha-t_{J-1}\right)\frac{\alpha-t_{J-1}}{(q-\alpha)(q-t_{J-1})} \\ &= \frac{a\varepsilon q}{(q-\alpha)^2}\cdot\frac{(\alpha-t_{J-1})^2}{q-t_{J-1}}. \end{aligned}$$

We now have

$$\begin{aligned} \varepsilon\frac{H-h_J}{h_JH} &= \frac{a\varepsilon^2 q}{(q-\alpha)^2}\cdot\frac{(\alpha-t_{J-1})^2}{q-t_{J-1}}\cdot\frac{q-\alpha}{a\varepsilon q}\cdot\frac{1}{\frac{t_J-\alpha}{q-\alpha}+\frac{\alpha-t_{J-1}}{q-t_{J-1}}}\cdot\frac{(q-\alpha)^2N}{a\varepsilon q} \\ &= \frac{(q-\alpha)N}{aq}\cdot\frac{(\alpha-t_{J-1})^2}{q-t_{J-1}}\cdot\frac{(q-\alpha)(q-t_{J-1})}{\frac{q}{N}-\alpha^2+2\alpha t_{J-1}-t_{J-1}t_J} \\ &= \frac{(q-\alpha)^2N}{aq}\cdot\frac{(\alpha-t_{J-1})^2}{\omega}\le\frac{\zeta^2\varepsilon}{aqN}\cdot\frac{1}{\omega}, \end{aligned}$$

where

$$\omega:=\frac{q}{N}-\alpha^2+2\alpha t_{J-1}-t_{J-1}t_J$$

and where in the last step we used (3.2) and the fact that $0\le\alpha-t_{J-1}\le 1/N$. The denominator ω can be estimated as follows:

$$\begin{aligned} \omega &= \frac{q}{N}-(\alpha-t_{J-1})^2-\frac{t_{J-1}}{N} \\ &= \frac{\zeta\sqrt{\varepsilon}+\alpha}{N}-(\alpha-t_{J-1})^2-\frac{t_{J-1}}{N} \\ &= \frac{\zeta\sqrt{\varepsilon}}{N}+\frac{1}{N}\left(\alpha-t_{J-1}\right)-\left(\alpha-t_{J-1}\right)^2 \\ &= \frac{\zeta\sqrt{\varepsilon}}{N}+\left(\alpha-t_{J-1}\right)\left(t_J-\alpha\right) \\ &\ge \frac{\zeta\sqrt{\varepsilon}}{N},\ \text{ since } \left(\alpha-t_{J-1}\right)\left(t_J-\alpha\right)\ge 0. \end{aligned}$$

Therefore,

$$\varepsilon\frac{H-h_J}{h_J H}\leq\frac{\zeta^2\varepsilon}{aqN}\cdot\frac{N}{\zeta\sqrt{\varepsilon}}=\frac{\zeta\sqrt{\varepsilon}}{aq}.$$

This completes the proof of (4.4). □

It is easy to see that $\tilde{A}_N$ is an L-matrix. The next lemma shows that $\tilde{A}_N$ is an M-matrix and that the modified discretization (4.2) is stable uniformly in ε.

Lemma 4.2. *Let ε be sufficiently small, independently of N, and let $a > 4/\beta$. Then the matrix $\tilde{A}_N$ of the system* (4.2) *satisfies*

$$\left\|\tilde{A}_N^{-1}\right\|\leq C.$$

Proof. We want to construct a vector $v=[v_0,v_1,\ldots,v_N]^T$ such that

(a) $v_i\geq\delta$, $i=0,1,\ldots,N$, where δ is a positive constant independent of both ε and N,
(b) $v_i\leq C$, $i=0,1,\ldots,N$,
(c) $\sigma_i:=l_iv_{i-1}+d_iv_i+r_iv_{i+1}\geq\delta$, $i=1,2,\ldots,N-1$.

Then, according to the M-criterion,

$$\|\tilde{A}_N^{-1}\|\leq\delta^{-1}\|v\|\leq C.$$

The following choice of the vector v is motivated by [7, 9, 17]:

$$v_i=\begin{cases}\alpha-Hi+\lambda, & i\leq J-1,\\ \alpha-Hi+\dfrac{\lambda}{1+\rho_J}(1+\rho)^{J-i}, & i\geq J,\end{cases}$$

where $\rho_J=\beta h_J/(2\varepsilon)$, $\rho=\beta H/(2\varepsilon)$, and α and λ are fixed positive constants. Since $HN\leq C$, there exists a constant α such that $v_i\geq\alpha-Hi\geq\delta>0$, so the condition (a) is satisfied. Then, because of $v_i\leq\alpha+\lambda$, the condition (b) holds true if we show that $\lambda\leq C$. We do this next as we verify the condition (c).

When $1\leq i\leq J-2$, we use (4.3) to get

$$\begin{aligned}\sigma_i&=(l_i+d_i+r_i)v_i+l_iH-r_iH\\&=\frac{\hbar_i}{H}c_iv_i-\frac{\varepsilon}{h_i}+\frac{\varepsilon}{h_{i+1}}+\frac{b_i\hbar_i}{h_{i+1}}\\&\geq-\left(\frac{\varepsilon}{h_i}-\frac{\varepsilon}{h_{i+1}}\right)+\frac{b_i}{2}+\frac{b_ih_i}{2h_{i+1}}\\&=-\frac{\varepsilon(h_{i+1}-h_i)}{h_ih_{i+1}}+\frac{b_i}{2}+\frac{b_ih_i}{2h_{i+1}}\\&\geq-\frac{2}{a}+\frac{b_i}{2}\geq\frac{\beta}{2}-\frac{2}{a}=:\delta>0.\end{aligned}$$

The constant δ exists because of the assumption $a>4/\beta$.

For $i = J-1$, we have

$$\begin{aligned}
\sigma_{J-1} &= \frac{\hbar_{J-1}}{H} c_{J-1} v_{J-1} + l_{J-1} H - r_{J-1} H \\
&\quad + \lambda l_{J-1} + \lambda d_{J-1} + r_{J-1} \frac{\lambda}{1+\rho_J} \\
&\geq -\frac{\varepsilon}{h_{J-1}} + \frac{\varepsilon}{h_J} + \frac{b_{J-1}\hbar_{J-1}}{h_J} - r_{J-1} \frac{\lambda \rho_J}{1+\rho_J} \\
&\geq -\frac{\varepsilon}{h_{J-1}} + \frac{b_{J-1}}{2} - r_{J-1} \frac{\lambda \rho_J}{1+\rho_J} \\
&\geq -\frac{\varepsilon}{h_{J-1}} + \frac{\beta}{2} + \left(\frac{\varepsilon}{h_J H} + \frac{b_{J-1}\hbar_{J-1}}{h_J H} \right) \frac{\lambda \beta h_J}{2\varepsilon + \beta h_J} \\
&= -\frac{\varepsilon}{h_{J-1}} + \frac{\beta}{2} + \left(\frac{2\varepsilon + b_{J-1}(h_{J-1} + h_J)}{2 h_J H} \right) \frac{\lambda \beta h_J}{2\varepsilon + \beta h_J} \\
&\geq \frac{\beta}{2} - \frac{\varepsilon}{h_{J-1}} + \frac{\lambda \beta}{4H} \geq \frac{\beta}{2} > \delta
\end{aligned}$$

with a suitable positive constant λ. We can choose such λ because the estimates $H \leq 2N^{-1}$ and $q - t_{J-1} \leq q - t_{J-2} \leq 1$ imply

$$\frac{\lambda \beta}{4H} - \frac{\varepsilon}{h_{J-1}} = \frac{\lambda \beta}{4H} - \frac{N}{aq} \left(q - t_{J-1}\right) \left(q - t_{J-2}\right) \geq N \left(\frac{\lambda \beta}{8} - \frac{1}{aq} \right) \geq 0.$$

For $i = J$, we get

$$\begin{aligned}
\sigma_J &= \frac{\hbar_J}{H} c_J v_J + l_J H - r_J H + \lambda \left[l_J + \frac{d_J}{1+\rho_J} + \frac{r_J}{(1+\rho_J)(1+\rho)} \right] \\
&\geq -\frac{\varepsilon}{h_J} + \frac{\varepsilon}{H} + \frac{b_J \hbar_J}{H} \\
&\quad + \frac{\lambda}{(1+\rho_J)(1+\rho)} \left[l_J (1+\rho_J)(1+\rho) + d_J(1+\rho) + r_J \right] \\
&\geq \frac{\varepsilon}{H} - \frac{\varepsilon}{h_J} + \frac{b_J}{2} \\
&\quad + \frac{\lambda}{(1+\rho_J)(1+\rho)} \left[l_J (1+\rho_J)(1+\rho) + d_J(1+\rho) + r_J \right] \\
&\geq \frac{\beta}{2} - \frac{\varepsilon(H - h_J)}{h_J H} \geq \delta > 0.
\end{aligned}$$

The above estimate holds true because (4.4) implies that

$$\frac{\varepsilon(H - h_J)}{h_J H} \leq \frac{\zeta \sqrt{\varepsilon}}{aq} \leq \frac{2}{a},$$

when ε is sufficiently small, and because we can show that

$$\left[l_J (1+\rho_J)(1+\rho) + d_J(1+\rho) + r_J \right] \geq 0.$$

Indeed,

$$\begin{aligned} l_J(1+\rho_J)(1+\rho)+d_J(1+\rho)+r_J &= l_J\rho_J + l_J\rho_J\rho - r_J\rho \\ &= -\frac{\varepsilon}{h_J H}\frac{\beta h_J}{2\varepsilon} - \frac{\varepsilon}{h_J H}\frac{\beta h_J}{2\varepsilon}\frac{\beta H}{2\varepsilon} \\ &\quad + \left[\frac{\varepsilon}{H^2} + \frac{b_J \hbar_J}{H^2}\right]\frac{\beta H}{2\varepsilon} \\ &= -\frac{\beta^2}{4\varepsilon} + \frac{\beta b_J \hbar_J}{2H\varepsilon} \\ &= -\frac{\beta^2}{4\varepsilon} + \frac{\beta b_J (h_J + H)}{4H\varepsilon} \\ &\geq -\frac{\beta^2}{4\varepsilon} + \frac{\beta b_J}{4\varepsilon} \geq 0. \end{aligned}$$

Finally, when $J+1 \leq i \leq N-1$, we have

$$\begin{aligned} \sigma_i &= c_i v_i + l_i H - r_i H + \frac{l_i}{1+\rho_J}\left[\frac{\lambda}{(1+\rho)^{i-1-J}} - \frac{\lambda}{(1+\rho)^{i-J}}\right] \\ &\quad + \frac{r_i}{1+\rho_J}\left[\frac{\lambda}{(1+\rho)^{i+1-J}} - \frac{\lambda}{(1+\rho)^{i-J}}\right] \\ &\geq b_i + \frac{\rho(1+\rho)l_i - \rho r_i}{(1+\rho_J)(1+\rho)^{i+1-J}}\lambda \\ &\geq \frac{\beta}{2} + \frac{(l_i - r_i + l_i\rho)\rho}{(1+\rho_J)(1+\rho)^{i+1-J}}\lambda \\ &= \frac{\beta}{2} + \left(\frac{b_i}{H} - \frac{\beta}{2H}\right)\frac{\lambda\rho(1+\rho)^{J-i-1}}{1+\rho_J} \\ &\geq \frac{\beta}{2} > \delta. \end{aligned}$$

□

By examining the elements of the matrix $\tilde{A}_N$, we see that

$$\|\tilde{A}_N\| \leq CN^2.$$

When we combined this with Lemma 4.2, we get the following result.

Theorem 4.3. *The matrix $\tilde{A}_N$ of the system (4.2) satisfies*

$$\kappa(\tilde{A}_N) \leq CN^2.$$

5 Uniform convergence and numerical experiments

Let τ_i, $i = 1, 2, \ldots, N-1$, be the consistency error of the finite-difference operator $\mathcal{L}^N$,

$$\tau_i = \mathcal{L}^N u_i - f_i.$$

We have

$$\tau_i = \tau_i[u] := \mathcal{L}^N u_i - (\mathcal{L}u)_i$$

and by Taylor's expansion we get that

$$|\tau_i[u]| \leq Ch_{i+1}(\varepsilon\|u'''\|_i + \|u''\|_i), \tag{5.1}$$

where $\|g\|_i := \max_{x_{i-1}\le x\le x_{i+1}} |g(x)|$ for any $C(I)$-function g. Let us define

$$\tilde{\tau}_i[u] = \begin{cases} \dfrac{\hbar_i}{H}\tau_i[u], & 1 \le i \le J, \\ \tau_i[u], & J+1 \le i \le N-1. \end{cases}$$

Lemma 5.1. *The following estimate holds true for all $i = 1, 2, \dots, N-1$:*

$$|\tilde{\tau}_i[u]| \le CN^{-1}.$$

Proof. We use the decomposition (2.1) and estimates (2.2). For the smooth part of the solution, it is easy to show that $|\tilde{\tau}[s]| \le CN^{-1}$. Then we need to show that

$$|\tilde{\tau}_i[y]| \le CN^{-1}.$$

Case 1. Let $i \ge J+1$, i.e. $t_{i-1} \ge t_J \ge \alpha$. Then we have

$$\begin{aligned} |\tilde{\tau}_i[y]| = |\tau_i[y]| &\le Ch_{i+1}\left(\varepsilon\|y'''\|_i + \|y''\|_i\right) \\ &\le CN^{-1}\lambda'(t_{i+1})\varepsilon^{-2}e^{-\beta\lambda(t_{i-1})/\varepsilon} \\ &\le CN^{-1}\lambda'(t_{i+1})\varepsilon^{-2}e^{-\beta\lambda(\alpha)/\varepsilon} \\ &\le CN^{-1}\varepsilon^{-2}e^{-a\beta\alpha/(\zeta\sqrt{\varepsilon})} \\ &\le CN^{-1}, \end{aligned}$$

where we have used the fact that $\varepsilon^{-2}e^{-a\beta\alpha/(\zeta\sqrt{\varepsilon})} \le C$.

Case 2. Let $i \le J$, i.e. $t_{i-1} < \alpha$, and at the same time, let $t_{i-1} \le q - 3/N$. Note that, when $t_{i-1} \le q - 3/N$, we have

$$t_{i+1} \le q - 1/N < q \quad \text{and} \quad q - t_{i+1} \ge \frac{1}{3}(q - t_{i-1}).$$

This is because

$$q - t_{i-1} \ge \frac{3}{N} \quad \Rightarrow \quad \frac{2}{3}(q - t_{i-1}) \ge \frac{2}{N},$$

which gives

$$q - t_{i+1} = q - t_{i-1} - \frac{2}{N} = \frac{1}{3}(q - t_{i-1}) + \frac{2}{3}(q - t_{i-1}) - \frac{2}{N} \ge \frac{1}{3}(q - t_{i-1}).$$

Therefore,

$$\begin{aligned} |\tilde{\tau}_i[y]| = \frac{\hbar_i}{H}|\tau_i[y]| &\le \frac{\hbar_i}{H}Ch_{i+1}\left(\varepsilon\|y'''\|_i + \|y''\|_i\right) \\ &\le CN^{-1}\left[\lambda'(t_{i+1})\right]^2\varepsilon^{-2}e^{-\beta\lambda(t_{i-1})/\varepsilon} \\ &\le CN^{-1}\left[\phi'(t_{i+1})\right]^2 e^{-a\beta\phi(t_{i-1})} \\ &\le C\varepsilon^{-1}N^{-1}(q - t_{i+1})^{-4}e^{-a\beta(q/(q-t_{i-1})-1)} \\ &\le CN^{-1}(q - t_{i-1})^{-4}e^{-a\beta q/(q-t_{i-1})} \\ &\le CN^{-1}, \end{aligned}$$

because $(q - t_{i-1})^{-4}e^{-a\beta q/(q-t_{i-1})} \le C$.

Case 3. In the last case, we consider the remaining possibility, $q - 3/N < t_{i-1} < \alpha$. We use the fact that $\mathcal{L}y = 0$ to work with

$$|\tilde{\tau}_i[y]| = \frac{h_i}{H}|\tau_i[y]| \leq \frac{\hbar_i}{H}\left(P_i + Q_i + R_i\right),$$

where

$$P_i = \varepsilon|D''y_i|, \quad Q_i = b_i|D'y_i|, \quad \text{and} \quad R_i = c_i|y_i|.$$

We now follow closely the technique in [16, Lemma 5], (see also [9, 17]), to get

$$\begin{aligned}
\frac{\hbar_i}{H}\left(P_i + Q_i + R_i\right) &\leq C\left[\frac{\hbar_i}{H}\left(\frac{1}{\hbar_i}\varepsilon \cdot 2\|y'\|_i\right) + \frac{\hbar_i}{H}\left(\frac{1}{h_{i+1}}\|y\|_i\right) + e^{-\beta\lambda(t_i)/\varepsilon}\right] \\
&\leq CNe^{-\beta\lambda(t_{i-1})/\varepsilon} \\
&\leq CNe^{-a\beta\phi(t_{i-1})} \\
&\leq CNe^{-a\beta\phi(q-3/N)} \\
&\leq CNe^{-a\beta(qN/3-1)} \\
&\leq CN^{-1}.
\end{aligned}$$

□

Remark 5.2. The technique used in the above proof is based on [15], where the same approach is successfully applied to reaction-diffusion problems. This approach is originally due to Bakhvalov [1]. The technique works here for convection-diffusion problems (1.1) because an extra ε-factor is obtained from the preconditioner (4.1).

When Lemmas 4.2 and 5.1 are combined, which amounts to the use of the consistency-stability principle, we obtain the following result.

Theorem 5.3. *Let ε be sufficiently small, independently of N, and let $a > 4/\beta$. Then the solution U^N of the discrete problem* (2.4) *on the VB-mesh satisfies*

$$\left\|U^N - u^N\right\| \leq CN^{-1},$$

where u is the solution of the continuous problem (1.1).

We conclude our paper by reporting a test problem taken from [4, page 1]:

$$-\varepsilon u'' - u' = 1, \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{5.2}$$

The exact solution of this problem is known. Tables 1 and 2 present the maximum point consistency errors without and with preconditioning, respectively, whereas Table 3 shows the errors in the max norm and the rate of convergence calculated from the maximum errors.

6 Declarations

Author's Contributions

The authors have contributed equally.

Conflicts of Interest

The authors declare no conflict of interest.

$-\log\varepsilon$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
1	5.94e-01	3.05e-01	1.55e-01	7.77e-02	3.90e-02	1.95e-02
2	5.94e+00	3.05e+00	1.55e+00	7.77e-01	3.90e-01	1.95e-01
3	5.94e+01	3.05e+01	1.55e+01	7.77e+00	3.90e+00	1.95e+00
4	5.94e+02	3.05e+02	1.55e+02	7.77e+01	3.90e+01	1.95e+01
5	5.94e+03	3.05e+03	1.55e+03	7.77e+02	3.90e+02	1.95e+02
6	5.94e+04	3.05e+04	1.55e+04	7.77e+03	3.90e+03	1.95e+03
7	5.94e+05	3.05e+05	1.55e+05	7.77e+04	3.90e+04	1.95e+04
8	5.94e+06	3.05e+06	1.55e+06	7.77e+05	3.90e+05	1.95e+05

Table 1: The maximum pointwise consistency error without preconditioning on the Vulanović-Bakhvalov mesh for the problem (5.2).

$-\log\varepsilon$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
1	4.01e-01	2.16e-01	1.12e-01	5.70e-02	2.88e-02	1.45e-02
2	3.10e-01	1.65e-01	8.50e-02	4.32e-02	2.18e-02	1.09e-02
3	2.61e-01	1.39e-01	7.16e-02	3.64e-02	1.83e-02	9.19e-03
4	2.46e-01	1.31e-01	6.75e-02	3.43e-02	1.73e-02	8.67e-03
5	2.41e-01	1.28e-01	6.62e-02	3.36e-02	1.69e-02	8.50e-03
6	2.40e-01	1.28e-01	6.58e-02	3.34e-02	1.68e-02	8.45e-03
7	2.39e-01	1.27e-01	6.57e-02	3.33e-02	1.68e-02	8.43e-03
8	2.39e-01	1.27e-01	6.56e-02	3.33e-02	1.68e-02	8.43e-03

Table 2: The maximum pointwise consistency error with preconditioning on the Vulanović-Bakhvalov mesh for the problem (5.2).

$-\log\varepsilon$	$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$	$N=2^{10}$
1	4.82e-02	2.54e-02	1.31e-02	6.63e-03	3.34e-03	1.68e-03
	0.92	0.96	0.98	0.99	0.99	–
2	7.43e-02	3.86e-02	1.97e-02	9.93e-03	4.99e-03	2.50e-03
	0.94	0.97	0.99	0.99	1.00	–
3	7.49e-02	3.87e-02	1.97e-02	9.93e-03	4.99e-03	2.50e-03
	0.95	0.97	0.99	0.99	1.00	–
4	7.49e-02	3.87e-02	1.97e-02	9.93e-03	4.99e-03	2.50e-03
	0.95	0.97	0.99	0.99	1.00	–
5	7.49e-02	3.87e-02	1.97e-02	9.93e-03	4.99e-03	2.50e-03
	0.95	0.97	0.99	0.99	1.00	–
6	7.49e-02	3.87e-02	1.97e-02	9.93e-03	4.99e-03	2.50e-03
	0.95	0.97	0.99	0.99	1.00	–
7	7.49e-02	3.87e-02	1.97e-02	9.93e-03	4.99e-03	2.50e-03
	0.95	0.97	0.99	0.99	1.00	–
8	7.49e-02	3.87e-02	1.97e-02	9.93e-03	4.99e-03	2.50e-03
	0.95	0.97	0.99	0.99	1.00	–

Table 3: The maximum pointwise errors and the calculated rate of convergence on the Vulanović-Bakhvalov mesh for the problem (5.2).

Funding

Not applicable.

Data Availability Statement

Not applicable.

Acknowledgements

We are grateful to the anonymous referee who helped us enhance the clarity of our work.

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