Research article

ISSN: 2980-1982
2025, Vol. 3, No. 1, 1–10

Letters on
Applied and Pure Mathematics

Recovering initial condition for composite fractional relaxation equations

Le Dinh Long [1, *], Ngo Ngoc Hung [2]

1. Faculty of Information Technology, Industrial University of Ho Chi Minh City, Vietnam
2. Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Vietnam

* Corresponding author

Abstract

A backward problem for composite fractional relaxation equations is considered with Caputo's fractional derivative. Based on a spectral problem, the representation of solutions is established. Next, we show the mildly ill-posedness in the Hadamard sense. Afterthat, we show the regularization solution by two regularization methods : the Landweber regularization method and the iterative method. The convergent rate between the exact solution and the regularized solution is provided, under the a priori parameter choice rule.

Keywords: backward problem, initial data, diffusion-wave equation, Caputo fractional derivative, ill-posed problem, regularization method
2020 Mathematics Subject Classification: 35K99, 47J06, 47H10, 35K05
Article history: Received 15 Dec 2024 ; Accepted 05 Mar 2025, Online 25 Mar 2025

1 Introduction

In this paper, let $\Omega \subset \mathbb{R}^N$, we consider a composite fractional relaxation equation with time fractional derivative

$$\partial_t \theta(x,t) + \partial_t^{\beta} \mathcal{A}\theta(x,t) + \theta(x,t) = \mathcal{G}(x,t), x \in \mathcal{D}, t \in (0,T], \tag{1.1}$$

where ∂_t^β is the Caputo's fractional derivative of order $0 < \beta < 1$. The operator $\mathcal{A}$ stands for the unbounded symmetric and uniformly elliptic operator with domain $\mathcal{H}_0^1(\mathcal{D}) \cap \mathcal{H}^2(\mathcal{D})$ defined by

$$\mathcal{A}\theta(x,t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{N} a_{ij}(x)\frac{\partial}{\partial x_j}\theta(x,t)\Big) + b(x)\theta(x,t), \tag{1.2}$$

where $a_{ij} = a_{ji}, i,j = 1,2,\ldots,N$, and there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \mu\sum_{i=1}^{N}\xi_i^2, x \in \bar{\Omega}, \xi_i \in \mathbb{R}^N, \tag{1.3}$$

$a_{ij} \in C^1(\overline{\mathcal{D}}), b \in C(\overline{\mathcal{D}}), b(x) \geq 0$ for all $x \in \overline{\mathcal{D}}$. Equation (1.1) satisfying the following terminal/boundary value conditions:

$$\theta(x,t) = 0, x \in \partial\mathcal{D}, t \in (0,T], \theta(x,T) = f(x), x \in \mathcal{D}, \tag{1.4}$$

where f is the terminal value status, $\mathcal{G}$ is a linear function, and T is a backward finite time which ensures that some situation of phenomena can be measured at point.

$$\|f_\epsilon - f\|_{L^2(\mathcal{D})} + \|\mathcal{G}_\epsilon - \mathcal{G}\|_{L^1(0,T;L^2(\mathcal{D}))} \leq \epsilon, \tag{1.5}$$

where $\epsilon > 0$ is the noise level. Composite fractional relaxation equations find applications across various fields due to their ability to model complex, memory-dependent, and non-local processes. In viscoelastic materials, these equations describe stress-strain relationships that exhibit memory effects, as detailed in [5]. They are also used in the study of dielectric and magnetic materials to model complex permittivity and permeability, with foundational work [6]. In biological systems, fractional relaxation equations help analyze the viscoelastic properties of tissues and diffusion processes[7]. For anomalous diffusion, which is characteristic of many porous media and financial markets [8]. These equations improve control systems by incorporating memory and hereditary properties[9]. In geophysics, they model seismic wave propagation in heterogeneous materials [10]. Signal processing applications, which require the analysis of signals with long-range dependence[11]. In electrochemical systems, these models are used to understand battery cycles and corrosion phenomena [12]. Heat transfer in materials with memory effects is another area of application [13]. Finally, in economics and finance, fractional relaxation equations describe market dynamics and financial time series with long-term dependencies [14]. These diverse applications highlight the broad utility of composite fractional relaxation equations in modeling complex systems.

The fractional relaxation equation in (1.1)-(1.4) with order $\frac{1}{2}$ is associated with the Basset problem, which is a well-known problem in fluid dynamics. The Basset problem involves the acceleration of a particle in a viscous fluid under the influence of gravity [15]. The well-posedness os this kind problem was studied by Ashyralyev [16]. Karczewska et al investigated the existence of mild, weak, and strong solutions of this stochastic equation [17]. On Banach spaces Lizama et al introduced the existence, maximal regularity, and L^p integrability of the solutions of this type of problem [18]. The direct and backward problem for a general fractional relaxation equation were studied by Bazhlekova in [19]. Azhar Ali Zafar used fractional operators with and without singular kernels and special functions to consider a composite fractional relaxation differential equation [20]. Fan et al obtained approximate controllability for semilinear composite fractional relaxation equations [21]. As far as we know, there has been relatively little research on inverse problem for composite fractional relaxation equations. In this work, we introduce two methods to regularize the backward problem for such equations.

This paper is organized as follows. In Section 2 gives some preliminaries that are needed throughout the paper. In Section 3, we show the sought solution of problem (1.1)-(1.4), and an example describes the mismatch of the problem and the intercept of the source function in space $L^2(\mathcal{D})$. Section 4, we present Landweber method. Finally, we introduce Iterative regularization method method to solve the problem (1.1)-(1.4) and show the convergent rate under a priori parameter choice rule.

2 Preliminaries and the mild solution

Since $\mathcal{A}$ is a symmetric uniformly elliptic operator, it generates the following spectral problem

$$\mathcal{A}e_m(x) = \lambda_m e_m(x), x \in \mathcal{D}; e_m(x) = 0, x \in \partial\mathcal{D}, m \in \mathbb{N}, \tag{2.1}$$

where $\{\lambda_m\}_{m=1}^{\infty}$ denotes the set of eigenvalues satisfying

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \leq \cdots, \tag{2.2}$$

and $\lim_{m\to\infty} \lambda_m = \infty$, the corresponding eigenfunctions $e_m \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)$ for every $m \in \mathbb{N}$. Then, the sequence $\{e_k\}_{k=1}^{\infty}$ is orthonormal basis in $L^2(\mathcal{D})$. For $\tau \geq 0$, the operator $\mathcal{A}^\tau$ possesses the following representation

$$\mathcal{A}^\tau u = \sum_{m=1}^{\infty} \lambda_m^{\frac{\tau}{2}} \langle u, e_m\rangle e_m(x), u \in \mathcal{H}(\mathcal{A}^\tau) = \Big\{ u \in L^2(\Omega) : \sum_{m=1}^{\infty} \lambda_m^{\tau} \left|\langle u, e_m\rangle\right|^2 < \infty \Big\}, \tag{2.3}$$

and $\mathcal{H}(\mathcal{A}^\tau)$ is a Hilbert space with the norm

$$\|\theta\|_{\mathcal{H}(\mathcal{A}^\tau)} = \Big(\sum_{m=1}^{\infty} \lambda_m^{2\tau} \left|\langle \theta, e_m\rangle\right|^2 \Big)^{\frac{1}{2}}. \tag{2.4}$$

We first suppose that the relevant direct problem to problem (1.1)-(1.4) has a solution θ, that is

$$\begin{cases} \partial_t\theta + \partial_t^\beta \mathcal{A}\theta + \theta = \mathcal{G}(x,t), x \in \mathcal{D}, t \in (0,T], \\ \theta|_{\partial\mathcal{D}} = 0, \\ \theta(x,0) = \theta_0(x), x \in \mathcal{D}, \end{cases} \tag{2.5}$$

has a solution θ which satisfies the form

$$\theta(x,t) = \sum_{m=1}^{\infty} \theta_m(t) e_m(x), \tag{2.6}$$

where $\theta_m(t), m \geq 1$ satisfy the following fractional Cauchy problem

$$\begin{cases} \theta_m'(t) + \lambda_m \mathcal{D}_t^\beta \theta_m(t) + \theta_m(t) = \mathcal{G}_m(t), \\ \qquad\qquad\qquad\qquad \theta_m(0) = \theta_{0m}, \end{cases} \tag{2.7}$$

where we set $\theta_m(t) = \langle \theta(\cdot,t), e_m\rangle, \mathcal{G}_m(t) = \langle \mathcal{G}(\cdot,t), e_m\rangle$ and $f_m = \langle f, e_m\rangle$. By applying $\mathcal{A}e_m = \lambda_m e_m$ for all $m \in \mathbb{N}$. Taking the Laplace transform into the first equation in (1.1), and using $\mathcal{L}\big[\partial_t^\beta \theta_m(t)\big](\xi) = \xi^\beta \widehat{\theta}_k(\xi) - \xi^{\beta-1}\theta_m(0)$, we have

$$\xi\widehat{\theta}_m(\xi) - \theta_{0m} + \lambda_m \xi^\beta \widehat{\theta}_m(\xi) - \lambda_m \xi^{\beta-1}\theta_{0m} + \widehat{\theta}_m = \widehat{\mathcal{G}}_m. \tag{2.8}$$

This means that

$$\widehat{\theta}_m(\xi) = \widehat{\mathcal{B}}_m(\xi)\theta_{0m} + \widehat{\mathcal{C}}_m(\xi)\widehat{\mathcal{G}}_m, \tag{2.9}$$

where

$$\widehat{\mathcal{B}}_m(\xi) = \frac{1+\lambda_m \xi^{\beta-1}}{\xi+\lambda_m \xi^{\beta}+1}, \widehat{\mathcal{C}}_m(\xi) = \frac{1}{\xi+\lambda_m \xi^{\beta}+1}. \tag{2.10}$$

See in [2], we received

$$\theta_m(t) = \mathcal{B}_m(t)\theta_{0m} + \int_0^t \mathcal{C}_m(t-s)\mathcal{G}_m(s)ds, \tag{2.11}$$

where

$$\mathcal{B}_m(t) = \frac{1}{\pi}\int_0^\infty e^{-rt} \frac{\lambda_m r^{\beta-1}\sin(\beta\pi)}{\left(-r+\lambda_m r^{\beta}\cos(\beta\pi)+1\right)^2 + \left(\lambda_m r^{\beta}\sin(\beta\pi)\right)^2} dr, \tag{2.12}$$

and

$$\mathcal{C}_m(t) = \frac{1}{\pi}\int_0^\infty e^{-rt} \frac{\lambda_m r^{\beta}\sin(\beta\pi)}{\left(-r+\lambda_m r^{\alpha}\cos(\beta\pi)+1\right)^2 + \left(\lambda_m r^{\beta}\sin(\beta\pi)\right)^2} dr. \tag{2.13}$$

Therefore, solution u has the form

$$\theta(x,t) = \sum_{m=1}^{\infty} \mathcal{B}_m(t)\theta_{0,m}e_m(x) + \sum_{m=1}^{\infty}\left(\int_0^t \mathcal{C}_m(t-s)\mathcal{G}_m(s)ds\right)e_m(x). \tag{2.14}$$

Now, let $t=T$ into (2.14), it yields that

$$\theta(x,t) = \sum_{m=1}^{\infty} \frac{\mathcal{B}_m(t)}{\mathcal{B}_m(T)}\left(f_m - \int_0^T \mathcal{C}_m(T-s)\mathcal{G}_m(s)ds\right)e_m(x) + \sum_{m=1}^{\infty}\left(\int_0^t \mathcal{C}_m(t-s)\mathcal{G}_m(s)ds\right)e_m(x). \tag{2.15}$$

Lemma 2.1. *[3] For $0<\zeta<1$ and $k\geq 1$, define $p_k(\zeta) = \sum_{i=0}^{k-1}(1-\zeta)^i$ and $r_k(\zeta) = 1-\zeta p_k(\zeta) = (1-\zeta)^k$. Then we have*

$$\begin{aligned} &p_k(\zeta)\zeta^{\mu} \leq k^{1-\mu}, \quad 0\leq\mu\leq 1,\\ &r_k(\zeta)\zeta^{\nu} \leq \theta_\nu(k+1)^{-\nu}, \text{ where } \theta_\nu = \begin{cases} 1, & 0\leq\nu\leq 1,\\ \nu^\nu, & \nu>1. \end{cases} \end{aligned} \tag{2.16}$$

Lemma 2.2. *[1] Let $\beta\in(0,1)$, the function $\mathcal{B}_m(t), m\in\mathbb{N}$, is continuous at $t\geq 0$ and has the following properties:*

- *$\mathcal{B}_m(0)=1, 0<\mathcal{B}_m(t)\leq 1, t\geq 0$,*
- *$\mathcal{B}_m(t)$ are completely monotone for $t\geq 0$,*
- *the inequalities hold*

$$\frac{\mathcal{E}_3}{\lambda_m} \leq \mathcal{B}_m(t) \leq 1 - \frac{\mathcal{E}_1\lambda_m}{2(1+\lambda_m^2)}\left(1-e^{-2t}\right), \text{ where } \mathcal{E}_3 = \frac{\lambda_1^2\sin(\beta\pi)e^{-T}}{3\beta\pi\left(\lambda_1^2+2\right)}. \tag{2.17}$$

Lemma 2.3. *[1] Let $\beta\in(0,1)$, the function $\mathcal{C}_m(t), m\in\mathbb{N}$ is continuous at $t\geq 0$ and has the following properties:*

- *$\mathcal{C}_m(0)=1, 0<\mathcal{C}_m(t)\leq 1$, for $t\geq 0$,*
- *$\mathcal{C}_m(t)$ are completely monotone for $t\geq 0$,*
- *the inequalities hold*

$$\frac{\mathcal{E}_1\lambda_m}{1+\lambda_m^2}e^{-2t} \leq \mathcal{C}_m(t) \leq \frac{\mathcal{E}_2}{1+\lambda_m t^{1-\beta}}, t\geq 0, \tag{2.18}$$

whereby

$$\mathcal{E}_1 = \frac{\sin(\beta\pi)}{2^{2\beta+1}\beta\pi}\left(2^{\beta}-1\right), \qquad \mathcal{E}_2 = \frac{\Gamma(1-\beta)}{\pi\sin(\beta\pi)}. \tag{2.19}$$

3 The ill-posed problem

Let $\langle f, \mathcal{G}\rangle = \langle e_m, 0\rangle$, it is easy to see that $\|f\|_2 = 1$. Thank to the property $\mathcal{B}_m'(t) = -\mathcal{C}_m(t)$ for each $m \in \mathbb{N}$, we have

$$\begin{aligned}\|u(\cdot,0)\|^2_{L^2(\mathcal{D})} &= \sum_{m=1}^{\infty} \frac{1}{|\mathcal{B}_m(T)|^2} |\langle f, e_m\rangle|^2 \\ &\geq \sum_{m=1}^{\infty} \frac{2+2\lambda_m^2}{2+2\lambda_m^2 - \mathcal{E}_1\left(1-e^{-2T}\right)\lambda_m} \left|\langle f, e_m\rangle\right|^2 \geq 1,\end{aligned} \tag{3.1}$$

then from Lemma 2.2. Therefore, the initial data may not measure on $\mathcal{D}$. Therefore, the backward problem of fractional relaxation equation is non-well-posed. Next, we obtain the following theorem about the conditional stability.

Theorem 3.1. *If $\Xi(x) = \theta(x,0)$ and $\Xi \in \mathcal{H}^{\frac{p}{2}}$ satisfies the a priori bound condition*

$$\|\Xi\|_{\mathcal{H}^{\frac{p}{2}}} := \left(\sum_{m=1}^{\infty} \lambda_m^p \left|\langle \Xi, e_m\rangle\right|^2\right)^{\frac{1}{2}} \leq \mathcal{M}, \tag{3.2}$$

where $\mathcal{M} > 0$ is a constant, we have

$$\|\Xi\|_{L^2} \leq \mathcal{C}\mathcal{M}^{\frac{2}{p+2}} \|\Phi\|_{L^2}^{\frac{p}{p+2}}. \tag{3.3}$$

where $\Phi = f + \mathcal{K}_m(T)$, $\mathcal{C} = \mathcal{D}_3^{-\frac{p}{p+2}}$ is a constant depending on β, T, p, and λ_1.

4 Landweber method and convergent rate

Next, we give the Landweber iterative regularization solution of $\Xi(x)$, with the operator equation $\Xi = (I - b\mathcal{K}^*\mathcal{K})\,\Xi + b\mathcal{K}^* f$ to replace the equation $\mathcal{K}\Xi = f$ and obtain the following iterative as follows:

$$\Xi^{0,\epsilon}(x) = 0, \Xi_\alpha^\epsilon(x) = (I - b\mathcal{K}^*\mathcal{K})\,\Xi_{\alpha-1}^\epsilon(x) + b\mathcal{K}^*\Xi^\epsilon(x), \quad \alpha = 1,2,3,\ldots, \tag{4.1}$$

where I is a unit operator; α is the iterative step number, also known as the regularization parameter; and b is called the relaxation factor and satisfies $0 < b < \|\mathcal{K}\|^{-2}$. Note that the operator $\mathcal{R}_\alpha : L^2 \to L^2$ is defined as

$$\mathcal{R}_\alpha := b\sum_{k=0}^{\alpha-1} (I - b\mathcal{K}^*\mathcal{K})^k \mathcal{K}^*, \alpha = 1,2,3,\ldots \tag{4.2}$$

By simple calculation, we have

$$\Xi_\alpha^\epsilon(x) = \mathcal{R}_\alpha f^\epsilon(x) = b\sum_{k=0}^{\alpha-1} (I - b\mathcal{K}^*\mathcal{K})^\alpha \mathcal{K}^* f^\epsilon(x). \tag{4.3}$$

Since $\mathcal{K}$ is a self-adjoint operator, applying the singular values of the operator $\mathcal{K}$ and formula (4.1), we obtain the Landweber regularization solution of the inverse problem (1.1) as follows:

$$\Xi_\alpha^\epsilon(x) = \sum_{m=1}^{\infty} \frac{1 - \left(1 - b|\mathcal{B}_m(T)|^2\right)^\alpha}{|\mathcal{B}_m(T)|} \langle f^\epsilon, e_m\rangle e_m(x), \tag{4.4}$$

Theorem 4.1. *Let $\Xi_\alpha^\epsilon(x)$ given by (2.14) be the Landweber iterative regularization solution of the exact solution (4.4). Suppose the priori condition (3.1) and the noise assumption (1.5) hold. Choosing the regularization parameter $\alpha = \lfloor b \rfloor$, where*

$$b = \left(\frac{\mathcal{M}}{\epsilon}\right)^{\frac{4}{p+2}}, \tag{4.5}$$

we have

$$\left\|\Xi_\alpha^\epsilon - \Xi\right\|_{L^2} \textit{ is of order } \epsilon^{\frac{p}{p+2}}. \tag{4.6}$$

where $\lfloor b \rfloor$ denotes the largest integer less than or equal to b and $\mathcal{X} := \sqrt{b}\mathcal{M}^{\frac{2}{p+2}}(2+2\mathcal{E}_3^2)^{\frac{1}{2}} + \left(\frac{p}{2b\mathcal{E}_3^2}\right)^{\frac{p}{4}}\mathcal{M}^{\frac{2}{p+2}}$ is a positive constant.

Proof. We have

$$\left\|\Xi_\alpha^\epsilon - \Xi\right\|_{L^2} \le \left\|\Xi_\alpha^\epsilon - \Xi_\alpha\right\|_{L^2} + \left\|\Xi_\alpha - \Xi\right\|_{L^2}. \tag{4.7}$$

From (4.7), we have

$$\begin{aligned}
\|\Xi_\alpha^\epsilon - \Xi_\alpha\|_{L^2} &= \Bigg\| \sum_{m=1}^{\infty} \frac{1-\left(1-b\left|\mathcal{B}_m(T)\right|^2\right)^\alpha}{\left|\mathcal{B}_m(T)\right|}\langle f^\epsilon, e_m\rangle e_m(x) \\
&\qquad - \sum_{m=1}^{\infty} \frac{1-\left(1-b|\mathcal{B}_m(T)|^2\right)^\alpha}{\left|\mathcal{B}_m(T)\right|}\langle f, e_m\rangle e_n(x)\Bigg\|_{L^2} \\
&= \Bigg\| \sum_{m=1}^{\infty} \frac{1-\left(1-b|\mathcal{B}_m(T)|^2\right)^m}{\left|\mathcal{B}_m(T)\right|^2}\langle f^\epsilon - f, e_m\rangle e_m(x)\Bigg\|_{L^2} \\
&\le \sup_{m\ge 1} \mathcal{A}(m)\epsilon(2+2\mathcal{E}_2^2)^{\frac{1}{2}}.
\end{aligned} \tag{4.8}$$

where $\mathcal{A}(m) := \frac{1-\left(1-b|\mathcal{B}_m(T)|^2\right)^\alpha}{\left|\mathcal{B}_m(T)\right|}$. Since $\mathcal{B}_m(T)$ is a singular value of the operator $\mathcal{K}$ and $0 < b < \|\mathcal{K}\|^{-2}$, we obtain $0 < b\mathcal{B}_m(T) < 1$. Thank you Bernoulli inequality, we have

$$1-\left(1-b\left|\mathcal{B}_m(T)\right|^2\right)^\alpha \le \sqrt{1-\left(1-b|\mathcal{B}_m(T)|^2\right)^\alpha} \le \sqrt{b\alpha}|\mathcal{B}_m(T)|, \tag{4.9}$$

then we obtain

$$\mathcal{A}(m) \le \sqrt{b\alpha}. \tag{4.10}$$

Further, we can obtain

$$\left\|\Xi_\alpha^\epsilon - \Xi_\alpha\right\|_{L^2} \le \sup_{m\ge 1} \mathcal{A}(m)\epsilon \le \sqrt{b\alpha}\epsilon\left(2+2\mathcal{E}_2^2\right)^{\frac{1}{2}}. \tag{4.11}$$

From (5.6), we have

$$\begin{aligned}
\|\Xi_\alpha - \Xi\|_{L^2} &= \Bigg\| \sum_{m=1}^{\infty} \frac{1-\left(1-b\left|\mathcal{B}_m(T)\right|^2\right)^\alpha}{\left|\mathcal{B}_m(T)\right|}\langle f^\epsilon - f, e_m\rangle e_m(x) - \sum_{m=1}^{\infty} \frac{1}{\left|\mathcal{B}_m(T)\right|}\langle f, e_m\rangle e_m(x)\Bigg\|_{L^2} \\
&= \Bigg\| \sum_{m=1}^{\infty} \frac{-\left(1-b\left|\mathcal{B}_m(T)\right|^2\right)^\alpha}{\left|\mathcal{B}_m(T)\right|}\langle f, e_m\rangle e_m(x)\Bigg\|_{L^2} \\
&= \Bigg\| \sum_{m=1}^{\infty} \left(1-b|\mathcal{B}_m(T)|^2\right)^\alpha \lambda_m^{-\frac{p}{2}}\lambda_m^{\frac{p}{2}}\langle \Xi, e_m\rangle e_m(x)\Bigg\|_{L^2} \le \sup_{m\ge 1} \mathcal{Z}(m)\mathcal{M},
\end{aligned} \tag{4.12}$$

where $\mathcal{Z}(m) := \left(1-b|\mathcal{B}_m(T)|^2\right)^{\alpha}\lambda_m^{-\frac{p}{2}}$. According to Lemma 2.2, we have $\mathcal{Z}(m) \le (1-\frac{b|\mathcal{E}_3|^2}{\lambda_m^2})^{\alpha}\lambda_n^{-\frac{p}{2}}$. Let $\mathcal{Z}(r) := \left(1-\frac{b|\mathcal{E}_3|^2}{r^2}\right)^{\alpha} r^{-\frac{p}{2}}, r := \lambda_n$. We can find that s_0 satisfies $\mathcal{Z}'(r_0) = 0$, it gives $s_0 = (\frac{b|\mathcal{E}_3|^2(4\alpha+p)}{p})^{\frac{1}{2}}$, then we have $\mathcal{Z}(r) \le \left(\frac{p}{2b\mathcal{E}_3^2}\right)^{\frac{p}{4}}(\alpha+1)^{-\frac{p}{4}}$. Hence, we have

$$\left\|\Xi_\alpha - \Xi\right\|_{L^2} \le \left(\frac{p}{2b\mathcal{E}_3^2}\right)^{\frac{p}{4}}(\alpha+1)^{-\frac{p}{4}}\mathcal{M}. \tag{4.13}$$

Combining (4.11)-(4.12), we obtain

$$\left\|\Xi_\alpha^\epsilon - \Xi\right\|_{L^2} \le \mathcal{X}\mathcal{M}^{\frac{2}{p+2}}\epsilon^{\frac{p}{p+2}}, \tag{4.14}$$

where $\mathcal{X} := \sqrt{b}\mathcal{M}^{\frac{2}{p+2}}(2+2\mathcal{E}_3^2)^{\frac{1}{2}} + \left(\frac{p}{2b\mathcal{E}_3^2}\right)^{\frac{p}{4}}\mathcal{M}^{\frac{2}{p+2}}$. The proof of this Theorem is complete. □

5 Iterative regularization method

Let the function $\theta_\alpha^{\epsilon,n}(x,t)$ be the solution of the following iterative regularization problem

$$\begin{cases} \partial_t\theta_\alpha^{\epsilon,n}(x,t) + \partial_t^\beta\mathcal{A}\theta_\theta^{\epsilon,n}(x,t) + \theta_\alpha^{\epsilon,n}(x,t) = \mathcal{G}^\epsilon(x,t), & x\in\mathcal{D}, t\in(0,T], \\ \theta_\alpha^{\epsilon,n}(x,0) = \Xi_\alpha^\epsilon(x), & x\in\mathcal{D}. \\ \theta_\alpha^{\epsilon,n}(x,t) = 0, & x\in\partial\mathcal{D}, t\in(0,T], \\ \theta_\alpha^{\epsilon,n}(x,T) = f^\epsilon(x) - \alpha\mathcal{A}^\tau\Xi_\alpha^{\epsilon,n}(x) + \alpha\mathcal{A}^\tau\Xi_\alpha^{\epsilon,n-1}(x), & x\in\mathcal{D}. \end{cases} \tag{5.1}$$

For $n = 1,2,3,\ldots$, and the initial value $\theta_\alpha^{\epsilon,0}(x,t) = 0$, where $\alpha > 0$ is a regularization parameter and k is the number of iterations. In case $n = 1$, this method is transformed into the generalized QBV method , see in [4]. We get

$$\begin{aligned} \theta_\alpha^{\epsilon,n}(x,T) &= \sum_{m=1}^{\infty}\left[\langle\Xi_\alpha^{\epsilon,n}, e_m\rangle\mathcal{B}_m(T) + \int_0^T \mathcal{C}_m(T-s)\langle\mathcal{G}^\epsilon(s), e_m\rangle ds\right] e_m(x) \\ &= \sum_{m=1}^{\infty}\left[\langle f_m^\epsilon, e_m\rangle - \alpha\lambda_m^\tau\langle\Xi_\alpha^{\epsilon,n}, e_m\rangle + \alpha\lambda_m^\tau\langle\Xi_\alpha^{\epsilon,n-1}, e_m\rangle\right] e_m(x). \end{aligned} \tag{5.2}$$

We can deduce that

$$\begin{aligned} \Xi_\alpha^{\epsilon,n}(x) &= \sum_{m=1}^{\infty}\left[\frac{\langle h^\epsilon, e_m\rangle}{\mathcal{B}_m(T)+\alpha\lambda_m^\tau} + \frac{\alpha\lambda_m^\tau\langle\Xi_\alpha^{\epsilon,n-1}, e_m\rangle}{\mathcal{B}_m(T)+\alpha\lambda_m^\tau}\right] e_m(x) \\ &= \sum_{m=1}^{\infty}\sum_{i=0}^{n-1}\left(\frac{\alpha\lambda_m^\tau}{\mathcal{B}_m(T)+\alpha\lambda_m^\tau}\right)^i \frac{\langle h^\epsilon, e_m\rangle}{\mathcal{B}_m(T)+\alpha\lambda_m^\tau} e_m(x) \\ &= \sum_{m=1}^{\infty}\left(1-\left(\frac{\alpha\lambda_m^\tau}{\mathcal{B}_m(T)+\alpha\lambda_m^\tau}\right)^n\right)\frac{\langle h^\epsilon, e_m\rangle}{\mathcal{B}_m(T)} e_m(x), \end{aligned} \tag{5.3}$$

where $\langle h^\epsilon, e_m\rangle = \langle f^\epsilon, e_m\rangle - \int_0^T \mathcal{C}_m(T-s)\langle\mathcal{G}^\epsilon(s), e_m\rangle ds$.

From the Lemma 2.2, we can find that $(\frac{\alpha\lambda_m^\tau}{\mathcal{B}_m(T)+\alpha\lambda_m^\tau})^n$ may tend to infinity when $0 < \mathcal{B}_m(T) < 1$, and thus this will not cause an error estimate in our calculations. Thus, we choice the regularized solution $\Xi_\alpha^{\epsilon,n}(x)$ as

$$\Xi_\alpha^{\epsilon,n}(x) = \sum_{m=1}^{\infty}\left(1-\left(\frac{\alpha\lambda_m^\tau}{|\mathcal{B}_m(T)|+\alpha\lambda_m^\tau}\right)^n\right)\frac{\langle h^\epsilon, e_m\rangle}{|\mathcal{B}_m(T)|} e_m(x). \tag{5.4}$$

For simplicity, let

$$\mathcal{S}^{n}_{\alpha,\tau}(\lambda_m) = 1 - \left(\frac{\alpha\lambda_m^{\tau}}{|\mathcal{B}_m(T)| + \alpha\lambda_m^{\tau}} \right)^{n}, \tag{5.5}$$

then we obtain

$$\Xi^{\epsilon,n}_{\alpha}(x) = \sum_{m=1}^{\infty} \mathcal{S}^{n}_{\alpha,\tau}(\lambda_m) \frac{\langle h^{\epsilon}, e_m \rangle}{|\mathcal{B}_m(T)|} e_m(x). \tag{5.6}$$

Lemma 5.1. *[4] For constants $\alpha > 0, \tau \geq 0, T > 0, n > 0, \lambda_m \geq 1$, we have*

$$\sup_{m\in\mathbb{N}} \frac{\mathcal{S}^{n}_{\alpha,\tau}(\lambda_m)}{|\mathcal{B}_m(T)|} \leq n\mathcal{C}_{\tau}\alpha^{-\frac{1}{1+\tau}}. \tag{5.7}$$

Lemma 5.2. *[4] For constants $\alpha > 0, \tau \geq 0, T > 0, k > 0$, we have*

$$\sup_{m\in\mathbb{N}} \left(1 - \mathcal{S}^{n}_{\alpha,\tau}(\lambda_m)\right)\lambda_m^{-\frac{p}{2}} \leq \begin{cases} \mathcal{E}_3^{-n}\alpha^{n}, & \dfrac{p}{2n} \geq 1+\tau, \\ \mathcal{E}_4^{n} n^{-n}\alpha^{\frac{p}{2(1+\tau)}}, & 0 < \dfrac{p}{2n} < 1+\tau. \end{cases} \tag{5.8}$$

Theorem 5.3. *Let the regularized solution $\Xi^{\epsilon,n}_{\alpha}(x)$ be given by (5.6). If the noise data f^{ϵ} satisfies (1.5) and the exact solution $\Xi(x)$ satisfies the a priori bound condition (3.1), we get*

- *If $\dfrac{p}{2n} \geq 1+\tau$ and choose $\alpha = \left(\frac{\delta}{E}\right)^{\frac{1+\tau}{1+k+k\tau}}$, we have*

$$\left\|\Xi^{\epsilon,n}_{\alpha} - \Xi\right\|_2 \textit{ is of order } \epsilon^{\frac{n+n\tau}{1+n+n\tau}}. \tag{5.9}$$

- *If $0 < \dfrac{p}{2n} < 1+\tau$ and choose $\alpha = \left(\frac{\epsilon}{\mathcal{M}}\right)^{\frac{2(1+\tau)}{p+2}}$, we have*

$$\left\|\Xi^{\epsilon,n}_{\alpha} - \Xi\right\|_2 \textit{ is of order } \epsilon^{\frac{p}{p+2}}. \tag{5.10}$$

Proof. We obtain

$$\left\|\Xi^{\epsilon,n}_{\alpha} - \Xi\right\|_{L^2} \leq \left\|\Xi^{\epsilon,n}_{\alpha} - \Xi^{n}_{\alpha}\right\|_{L^2} + \left\|\xi^{n}_{\alpha} - \Xi\right\|_{L^2}. \tag{5.11}$$

First of all, we have estimate

$$\left\|h^{\epsilon} - h\right\|_2^2 \leq \sum_{m=1}^{\infty} \left(f^{\epsilon} - f + \int_0^T \mathcal{C}_m(T-s)\left(\mathcal{G}^{\epsilon}(s) - \mathcal{G}(s)\right)ds \right)^2 \leq \epsilon^2\left(2 + 2\mathcal{E}_2^2\right). \tag{5.12}$$

From (5.4) and Lemma 5.1, we have

$$\begin{aligned} \left\|\Xi^{\epsilon,n}_{\alpha} - \Xi^{n}_{\alpha}\right\|_{L^2} &= \left\| \sum_{m=1}^{\infty} \mathcal{S}^{n}_{\alpha,\tau}(\lambda_m) \frac{1}{|\mathcal{B}_m(T)|} \langle h^{\epsilon} - h, e_m\rangle e_m(x) \right\|_{L^2} \\ &= \left\| \sum_{m=1}^{\infty} \mathcal{S}^{n}_{\alpha,\tau}(\lambda_m) \frac{1}{|\mathcal{B}_m(T)|} \langle h^{\epsilon} - h, e\rangle e(x) \right\|_{L^2} \\ &\leq \sup_{m\in\mathbb{N}} \frac{\mathcal{S}^{n}_{\alpha,\tau}(\lambda_m)}{|\mathcal{B}_m(T)|} \left\| \sum_{m=1}^{\infty} \langle h^{\epsilon} - h, e_m\rangle e_m(x) \right\|_{L^2} \\ &\leq n\mathcal{C}_{\tau}\alpha^{-\frac{1}{1+\tau}}\epsilon\left(2 + 2\mathcal{E}_2^2\right)^{\frac{1}{2}}. \end{aligned} \tag{5.13}$$

The Lemma 5.2 gives us the following

$$\begin{aligned}
\left\|\Xi_\alpha^n - \xi\right\|_{L^2} &= \Big\| \sum_{m=1}^{\infty} \left(\mathcal{S}_{\alpha,\tau}^n(\lambda_m) - 1\right) \frac{\langle h, e_m \rangle}{|\mathcal{B}_m(T)|} e_m(x) \Big\|_{L^2} \\
&= \Big\| \sum_{m=1}^{\infty} \left(1 - \mathcal{S}_{\alpha,\tau}^n(\lambda_m)\right) \lambda_m^{-\frac{p}{2}} \lambda_m^{\frac{p}{2}} \Xi e_m(x) \Big\|_{L^2} \\
&\le \sup_{m\in\mathbb{N}} \left(1 - \mathcal{S}_{\alpha,\tau}^n(\lambda_m)\right) \lambda_m^{-\frac{p}{2}} \Big\| \sum_{m=1}^{\infty} \lambda_m^{\frac{p}{2}} \Xi_m e_m(x) \Big\|_{L^2} \\
&\le \mathcal{M} \sup_{m\in\mathbb{N}} \left(1 - \mathcal{S}_{\alpha,\tau}^n(\lambda_m)\right) \lambda_m^{-\frac{p}{2}} \\
&\le \begin{cases} \mathcal{E}_3^{-n} \alpha^n \mathcal{M}, & \frac{p}{2n} \ge 1+\tau, \\ \mathcal{E}_4^n n^{-n} \alpha^{\frac{p}{2(1+\tau)}} \mathcal{M}, & 0 < \frac{p}{2n} < 1+\tau. \end{cases}
\end{aligned} \tag{5.14}$$

Combining (5.13) and (5.14), we obtain

$$\left\|\Xi_\alpha^{\epsilon,n} - \Xi\right\|_{L^2} \le n\mathcal{C}_\tau \alpha^{-\frac{1}{1+\tau}} \epsilon \left(2 + 2\mathcal{E}_2^2\right)^{\frac{1}{2}} + \begin{cases} \mathcal{E}_3^{-n} \alpha^n \mathcal{M}, & \frac{p}{2n} \ge 1+\tau, \\ \mathcal{E}_4^n n^{-n} \alpha^{\frac{p}{2(1+\tau)}} \mathcal{M}, & 0 < \frac{p}{2n} < 1+\tau. \end{cases} \tag{5.15}$$

- If $\frac{p}{2n} \ge 1+\tau$, and $\alpha = \left(\frac{\epsilon}{\mathcal{M}}\right)^{\frac{1+\tau}{1+n+n\tau}}$, then we have

$$\left\|\Xi_\alpha^{\epsilon,n} - \Xi\right\|_{L^2} \le \left(n\mathcal{C}_\tau + \mathcal{E}_3^{-n}\right) \mathcal{M}^{\frac{1}{1+n+n\tau}} \epsilon^{\frac{n+n\tau}{1+n+n\tau}}, \tag{5.16}$$

- If $0 < \frac{p}{2n} < 1+\tau$, and $\alpha = \left(\frac{\epsilon}{\mathcal{M}}\right)^{\frac{2(1+\tau)}{p+2}}$, then we have

$$\left\|\Xi_\alpha^{\epsilon,n} - \Xi\right\|_{L^2} \le \left(n\mathcal{C}_\tau + \mathcal{E}_4^n n^{-n}\right) \mathcal{E}^{\frac{2}{p+2}} \epsilon^{\frac{p}{p+2}}. \tag{5.17}$$

□

6 Conclusion

In this paper, we investigate the problem of recovering initial conditions for the considered fractional expansion equations with Caputo fractional derivatives. The solution of the problem is established. This is an ill-formed problem in the Hadamard sense, with two regularization methods: Landweber and iterative methods. The error with the choice of a priori regularization parameters is shown.

7 Declarations

Funding

This research received no external funding.

Competing Interests

The authors declare no conflict of interest.

Ethical Approval

Not applicable

Authors's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Data Availability Statement

Not applicable.

Acknowledgements

The authors would like to thank the editors and reviewers for their time, effort, and valuable comments that led to possible improvements to the current version of our manuscript.

References

[1] Y Q Wu, J H Wei, *Regularization of a backward problem for composite fractional relaxation equations*, Mathematical Methods in the Applied Sciences, **4**6(1) (2023), pp. 180-196.

[2] R Gorenflo, F Mainardi, *Fractional Calculus: integral and differential equations of fractional order*, Mathematics, **49**(2), (2018), pp.277-290

[3] D G Li, J Fu, F Yang, X X Li, *Landweber iterative regularization method for identifying the initial value problem of the Rayleigh–Stokes equation*, Fractal and Fractional, **5**(4) (2021), 193.

[4] X Xiong, X Xue, Z Qian, Z, *A modified iterative regularization method for ill-posed problems*. Applied Numerical Mathematics, **122**, (2017) 108-128.

[5] F Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, 2010. World Scientific Publishing.

[6] S Havriliak, S J Havriliak, *Dielectric and Mechanical Relaxation in Materials: Analysis, Interpretation, and Application to Polymers*. Hanser/Gardner Publications, 1997.

[7] R L Magin *Fractional Calculus in Bioengineering*. Begell House Publishers, 2006.

[8] R Metzler, J Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*. Physics Reports, 339(1), (2000) 1-77.

[9] I Podlubny, *Fractional Differential Equations*. Academic Press, 1999.

[10] J M Carcione, *Wave Fields in Real Media: Wave Propagation in Anisotropic, Anelastic, Porous, and Electromagnetic Media*. Elsevier, (2014.)

[11] X Li, Y Zhao,*Fractional order signal processing: a new paradigm*. Signal Processing, **8**8(10), (2008), pp 2346-2357.

[12] E Barsoukov,J R Macdonald, *Impedance Spectroscopy: Theory, Experiment, and Applications*. John Wiley & Sons, 2018.

[13] Y Povstenko, *Fractional Thermoelasticity*. Springer, 2015.

[14] E Scalas, R Gorenflo, F Mainardi, *Fractional calculus and continuous-time finance*. Physica A: Statistical Mechanics and its Applications, **2**84(1-4), (2000), pp. 376-384.

[15] F Mainardi F, *Fractional calculus: some basic problems in continuum and statistical mechanics*. Fractals and Fractional Calculus in Continuum Mechanics. Vienna, Springer; (1997):291-348.

[16] A Ashyralyev, *Well-posedness of the Basset problem in spaces of smooth functions*. Appl Math Lett., **2**4 (2011) :1176-1180.

[17] A Karczewska, C Lizama, *Solutions to stochastic fractional relaxation equations*. Phys Scr T., **136**, (2009) :014030. 7pp.

[18] C Lizama, H Prado, *Fractional relaxation equations on Banach spaces*. Appl Math Lett, **2**3, (2010) :137-142.

[19] E Bazhlekova, *Estimates for a general fractional relaxation equation and application to an inverse source problem*, Math Methods Appl Sci, **4**1(18) (2018) 9018-9026. doi:10.1002/mma.4868.

[20] Azhar Ali Zafar, Jan Awrejcewicz, Olga Mazur & Muhammad Bilal Riaz, *Study of composite fractional relaxation differential equation using fractional operators with and without singular kernels and special functions*. Advances in Continuous and Discrete Models, https://doi.org/10.1186/s13662-021-03227-w.

[21] Z Fan, Q Dong, G. Li, *Approximate controllability for semilinear composite fractional relaxation equations*. Frac Calc Appl Anal, **1**9(1), (2016), pp. :267-284.