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On an ill-posed problem for system of coupled sinh-Gordon equations

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Abstract. The aim of this paper is considering the initial value problem for a system of coupled nonlinear sinh-Gordon equations by the association between two regularization methods: filter and truncation Fourier. Firstly, we give an example to show that the problem does not satisfy the third property which is called ill-posed in the sense of Hadamard. Secondly, under some a priori assumptions, we propose the stable regularization methods to regularize the system, i.e. the corresponding regularized solution converge to the exact solution in L^2-norm. Finally, to illustrate the proposed efficiency in the theoretical part, we show some numerical tests to check the convergence of the regularized solution and the regularized errors.

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1. Introduction

Let $\mathcal{H}$ be a Hilbert space, T be a positive number, denoting $\mathcal{T} := (0, T)$ and $\mathcal{D}$ be an open, bounded and connected domain in $\mathbb{R}^d, (d \geq 1)$ with a smooth boundary $\partial\mathcal{D}$. We consider the problem of finding a couple of unknown functions (x_1, x_2) which is called sought solution of the system of coupled sinh-Gordon equations considered as the following form

$$\begin{cases} \dfrac{\partial^2 x_1}{\partial t^2} = \alpha_1 \dfrac{\partial^2 x_1}{\partial s^2} + \beta_1 \sinh(\gamma_1 x_1 + \delta_1 x_2) + f_1(x_1, x_2) + F_1(s,t), & (s,t) \in \mathcal{D} \times \mathcal{T}, \\ \dfrac{\partial^2 x_2}{\partial t^2} = \alpha_2 \dfrac{\partial^2 x_2}{\partial s^2} + \beta_2 \sinh(\gamma_2 x_1 + \delta_2 x_2) + f_2(x_1, x_2) + F_2(s,t), & (s,t) \in \mathcal{D} \times \mathcal{T}, \end{cases} \tag{1}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are positive physical constants, the functions f_i are globally Lipschitz and F_i are giving functions which are called forcing functions for $i \in \{1, 2\}$. The Problem (1) under the homogeneous Dirichlet boundary condition as follows

$$x_1(s,t) = x_2(s,t) = 0 \text{ for } (s,t) \in \partial\mathcal{D} \times \mathcal{T}, \tag{2}$$

and associated the initial condition which is giving by

$$\begin{cases} x_1(s,t) = \phi_1(s) \text{ and } \dfrac{\partial}{\partial t} x_1(s,t) = \varphi_1(s), & (s,t) \in \mathcal{D} \times \{0\}, \\ x_2(s,t) = \phi_2(s) \text{ and } \dfrac{\partial}{\partial t} x_2(s,t) = \varphi_2(s), & (s,t) \in \mathcal{D} \times \{0\}. \end{cases} \tag{3}$$

The sine-Gordon equation is a type of nonlinear equation with partial derivatives that involves the d'Alembert operator and the sine of an unknown function as its two variables. It applies to one spatial and one temporal dimension. This equation was first derived by Bour (1862) for studying curved surfaces with negative curvature in 3-dimensional space, and later by Frenkel and Kontorova (1939) for studying how crystals have defects, which is called the Frenkel-Kontorova model. This equation became very popular in the 1970s because it has soliton solutions, which are special waves that do not change shape or speed.

Our main purpose in this paper consider the ill-posed problem (in the sense of Hadamard) for this system, after that we give examples to illustrate the problem. This paper has four main sections. In Section 2, we review some basic concepts and symbols that we need for the main result. Then, in Section 3, we present our main results, which include an example of a solution that is not well-defined. We also show how to make the solution well-defined and how to find it. In the last section, we give a numerical example to show how our method works in practice.

2. Some settings and notation

We start by defining the norms of some spaces in the following way. Firstly, let $\langle \cdot, \cdot \rangle_{\mathscr{L}^2}$ be the scalar product in $\mathscr{L}^2$ for all $g, h \in \mathscr{L}^2(\mathcal{D})$, we have

$$\langle f, h \rangle_{\mathscr{L}^2(\mathcal{D})} = \int_{\mathcal{D}} f(s)h(s)ds. \tag{4}$$

The notation $\|\cdot\|$ stands for the norm in the Banach space $\mathcal{B}$. Let us define the function space $(\mathscr{L}^2(\mathcal{D}))^2 := \mathscr{L}^2(\mathcal{D}) \times \mathscr{L}^2(\mathcal{D})$ with the usual inner product as follows

We use $\|\cdot\|$ to mean the norm in the Banach space $\mathcal{B}$. We also define the function space $(\mathscr{L}^2(\mathcal{D}))^2 := \mathscr{L}^2(\mathcal{D}) \times \mathscr{L}^2(\mathcal{D})$ with the usual inner product as follows

$$\langle V, W \rangle_{(\mathscr{L}^2(\mathcal{D}))^2} = \langle v_1, w_1 \rangle_{\mathscr{L}^2(\mathcal{D})} + \langle v_2, w_2 \rangle_{\mathscr{L}^2(\mathcal{D})}, \tag{5}$$

where $V = (v_1, v_2)$ and $W = (w_1, w_2)$ for all $v_1, v_2, w_1, w_2 \in \mathscr{L}^2(\mathcal{D})$ with the norm

$$\|V\|_{(\mathscr{L}^2(\mathcal{D}))^2} = \|v_1\|_{\mathscr{L}^2(\mathcal{D})} + \|v_2\|_{\mathscr{L}^2(\mathcal{D})}. \tag{6}$$

Moreover, we denote that $(\mathscr{L}^\infty([0,T]; \mathscr{L}^2(\mathcal{D})))^2 = \mathscr{L}^\infty([0,T]; \mathscr{L}^2(\mathcal{D})) \times \mathscr{L}^\infty([0,T]; \mathscr{L}^2(\mathcal{D}))$.

For $v \in \mathscr{C}([0,T]; \mathscr{L}^2(\mathcal{D}))$, let us define

$$\|v\|_{\mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D}))} = \sup_{0 \le t \le T} \|v(\cdot, t)\|_{\mathscr{L}^2(\mathcal{D})}.$$

Next, by denoting $(\mathscr{C}([0,T]; \mathscr{L}^2(\mathcal{D})))^2 = \mathscr{C}([0,T]; \mathscr{L}^2(\mathcal{D})) \times \mathscr{C}([0,T]; \mathscr{L}^2(\mathcal{D}))$, let α be a positive constant, we define a norm as follows

$$\|V\|_{(\mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D})))^2} = \sup_{0 \le t \le T} \left\{ \alpha^{\frac{t}{T}} \left(\|v_1(t)\|_{\mathscr{L}^2(\mathcal{D})} + \|v_2(t)\|_{\mathscr{L}^2(\mathcal{D})} \right) \right\}, \tag{7}$$

for any $V = (v_1, v_2) \in (\mathscr{C}([0,T]; \mathscr{L}^2(\mathcal{D})))^2$ and $v_1, v_2 \in \mathscr{C}([0,T]; \mathscr{L}^2(\mathcal{D}))$.

Finally, to fulfill this section, we give some assumptions as follows

[A$_1$] Suppose that the forcing terms (F_1, F_2) belong to $(\mathscr{L}^\infty([0,T]; \mathscr{L}^2(\mathcal{D})))^2$ and the functions f_i are globally Lipschitz source satisfying

$$\|f_i(U) - f_i(V)\|_{\mathscr{L}^2(\mathcal{D})} \le L_i \|U - V\|_{\mathscr{L}^2(\mathcal{D}) \times \mathscr{L}^2(\mathcal{D})}, \quad \text{for } i = 1, 2, \tag{8}$$

where $U = (x_1, x_2),\ V = (y_1, y_2)$ and L_i are constants independent of x_1, x_2, y_1, y_2.

[A$_2$] The data $X_0 = (\phi_1, \phi_2), \dot{X}_0 = (\varphi_1, \varphi_2)$ and $\widetilde{F} = (F_1, F_2)$ are noisy and are represented by the observation data $X_0^{\rm obs} = (\phi_1^{\rm obs}, \phi_2^{\rm obs}), \dot{X}_0^{\rm obs} = (\varphi_1^{\rm obs}, \varphi_2^{\rm obs})$ and $\widetilde{F}^{\rm obs} = (F_1^{\rm obs}, F_2^{\rm obs})$ satisfying

$$\|X_0 - X_0^{\rm obs}\|_{(\mathscr{L}^2(\mathcal{D}))^2} \le \epsilon,\ \|\dot{X}_0 - \dot{X}_0^{\rm obs}\|_{(\mathscr{L}^2(\mathcal{D}))^2} \le \epsilon, \tag{9}$$

$$\|\widetilde{F} - \widetilde{F}^{\rm obs}\|_{(\mathscr{L}^\infty([0,T];\mathscr{L}^2(\mathcal{D})))^2} \le \epsilon, \tag{10}$$

where ϵ is a noisy level constant which satisfies $\epsilon \to 0^+$.

3. Main result

In this section, we present three results as follows. In first subsection, we show a mild solution of the Problem (1)-(3) base on some conditions. In second subsection, we give an example to show that the Problem (1)-(3) does not satisfy the third property of definition of ill-posed problem which is given in the Hadamard sense. In next two subsections, we show two regularized methods of Problem (1)-(3).

Before giving the main result, we introduce an orthonormal eigenbasis $\{\xi_p\}_{p\in\mathbb{N}}$ associated with the eigenvalue $\{\ell_p\}_{p\in\mathbb{N}}$ in $\mathscr{L}^2(\mathcal{D})$. Noting that in the system (1), the orthonormal eigenbasis and eigenvalues will be $\{\xi_{i,p}\}_{p\in\mathbb{N}}$ and $\{\ell_{i,p}\}_{p\in\mathbb{N}}$, for $i \in \{1,2\}$, respectively.

In addition, we define the following Dirichlet-Laplacian

$$\mathcal{A}f := -\Delta f = -\sum_{n=1}^{d} \frac{\partial^2 f}{\partial s_n^2},\ \text{and}\ D(\mathcal{A}) = \mathscr{L}^2(\mathcal{D}) \cap H_0^1(\mathcal{D}). \tag{11}$$

Then we have the following lemma.

Lemma 3.1. There exist the orthonormal eigenbasis $\{\xi_{i,p}\}_{p\in\mathbb{N}} \in \mathbb{R}$ associated with the eigenvalue $\{\ell_{i,p}\}_{p\in\mathbb{N}}$ in $\mathscr{L}^2(\mathcal{D})$ for $i \in \{1,2\}$ such that

i) $\mathcal{A}\xi_{i,p} = \ell_{i,p}\xi_{i,p}$;

ii) $0 < \ell_{i,1} \le \ell_{i,2} \le \dots \le \ell_{i,p} \le \dots$ and $\lim_{p\to\infty} \ell_{i,p} \to \infty$.

iii) For $f \in \mathscr{L}^2(\mathcal{D})$, we have

$$f = \lim_{\mathcal{P}\to\infty} \sum_{p=1}^{\mathcal{P}} \langle f, \xi_{i,p}\rangle_{\mathscr{L}^2(\mathcal{D})} \xi_{i,p}$$

and for each $n, m \in \mathbb{N}$, we obtain

$$\langle \xi_{i,n}, \xi_{i,m}\rangle = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \ne m. \end{cases}$$

Proof. We can find the proof of Lemma 3.1 in [5, Section 6.5, Theorem 1]. □

Remark 3.1. In $\mathscr{L}^2(0,\pi)$, the orthonormal eigenbasis is giving by $\xi_p = \sqrt{\frac{2}{\pi}}\sin(ps)$ and the eigenvalue can be $\ell_p = p^2,\ p = 1, 2, ...$

3.1. Mild solution of Problem (1)-(3)

By using the separate variable method and interpretation in a Fourier series form, we obtain the exact solution of Problem (1) in the following theorem.

Theorem 3.2. *Let $(\phi_1, \varphi_1), (\phi_2, \varphi_2) \in (\mathscr{L}(\mathcal{D}))^2$ and $(F_1, F_2) \in (\mathscr{L}^\infty([0,T]; \mathscr{L}^2(\mathcal{D})))^2$, then there exists a mild solution $(x_1, x_2) \in (\mathscr{C}([0,T]; \mathscr{L}^2(\mathcal{D})))^2$ which satisfying the integral equations as follows*

$$\begin{aligned} x_i(s,t) = \lim_{\mathcal{P}\to\infty} \sum_{p=1}^{\mathcal{P}} x_{i,p}(\cdot,t)\xi_{i,p}(s) = \lim_{\mathcal{P}\to\infty} \sum_{p=1}^{\mathcal{P}} \Bigg[&\cosh\left(\alpha_i^{1/2}\ell_{i,p}^{1/2} t\right) \langle \phi_i(\cdot), \xi_{i,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+ \alpha_i^{1/2}\ell_{i,p}^{-1/2}\sinh\left(\alpha_i^{1/2}\ell_{i,p}^{1/2} t\right) \langle \varphi_i(\cdot), \xi_{i,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+ \int_0^t \alpha_i^{1/2}\ell_{i,p}^{-1/2}\sinh\left(\ell_{i,p}^{1/2}(t-z)\right) \Big\langle \beta_i \sinh(\gamma_i x_1(\cdot,z) + \delta_i x_2(\cdot,z)) \\ &+ f_i(x_1(\cdot,z), x_2(\cdot,z)) + F_i(\cdot,z), \xi_{i,p}(\cdot,z)\Big\rangle_{\mathscr{L}^2(\mathcal{D})} \mathrm{d}z \Bigg] \xi_{i,p}(s) \end{aligned} \tag{12}$$

where $x_{i,p}(t) = \langle x_i(\cdot,t), \xi_{i,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})}$ stand for its Fourier coefficient for $i \in \{1,2\}$.

Proof. The proof of Theorem 3.2 is divided into two steps.

Step 1. Since we have the set of eigenbases $\{\xi_{i,p}\}_{p\in\mathbb{N}}^{i=1,2}$, it is capable to present the solution of Problem (1) by Fourier series, i.e., $x_i(s,t) = \lim\limits_{\mathcal{P}\to\infty} \sum_{p=1}^{\mathcal{P}} x_{i,p}(\cdot,t)\xi_{i,p}(s) := \lim\limits_{\mathcal{P}\to\infty} \sum_{p=1}^{\mathcal{P}} x_{i,p}(t)\xi_{i,p}(s)$, we find the Fourier coefficient $x_{i,p}$. Indeed, by using the separating variables method, we transform the Problem (1) into a couple of differential equation as follows

$$\begin{cases} \lim\limits_{\mathcal{P}\to\infty} \sum\limits_{p=1}^{\mathcal{P}} x''_{1,p}(t)\xi_{1,p} = \alpha_1 \lim\limits_{\mathcal{P}\to\infty} \sum\limits_{p=1}^{\mathcal{P}} \ell_{1,p}^2 x_{1,p}(t)\xi_{1,p} + \lim\limits_{\mathcal{P}\to\infty} \sum\limits_{p=1}^{\mathcal{P}} \mathcal{R}_{1,p}(t, x_{1,p}(t), x_{2,p}(t))\xi_{1,p}, \\ \lim\limits_{\mathcal{P}\to\infty} \sum\limits_{p=1}^{\mathcal{P}} x''_{2,p}(t)\xi_{2,p} = \alpha_2 \lim\limits_{\mathcal{P}\to\infty} \sum\limits_{p=1}^{\mathcal{P}} \ell_{2,p}^2 x_{2,p}(t)\xi_{2,p} + \lim\limits_{\mathcal{P}\to\infty} \sum\limits_{p=1}^{\mathcal{P}} \mathcal{R}_{2,p}(t, x_{1,p}(t), x_{2,p}(t))\xi_{2,p}, \end{cases}$$

where for $i \in \{1,2\}$, we define

$$\begin{aligned} &\mathcal{R}_{i,p}(t, x_{1,p}(t), x_{2,p}(t)) \\ =& \Big\langle \beta_i \sinh(\gamma_i x_1(\cdot,t) + \delta_i x_2(\cdot,t)) + f_i(x_1(\cdot,t), x_2(\cdot,t)) + F_i(\cdot,t), \xi_{i,p}(\cdot)\Big\rangle_{\mathscr{L}^2(\mathcal{D})}. \end{aligned}$$

Then, for $t \in \mathcal{T}$, we have

$$\begin{cases} x''_{1,p}(t) = \alpha_1 \ell_{1,p}^2 x_{1,p}(t) + \mathcal{R}_{1,p}(t, x_{1,p}(t), x_{2,p}(t)), \\ x''_{2,p}(t) = \alpha_2 \ell_{2,p}^2 x_{2,p}(t) + \mathcal{R}_{2,p}(t, x_{1,p}(t), x_{2,p}(t)). \end{cases} \tag{13}$$

Step 2. Solving the system (13) to find the solution $(x_{1,p}, x_{2,p})$. To do this, multiplying the each equation of (13) to $\ell_{i,p}^{-1/2}\sinh\left(\ell_{i,p}^{1/2}(t-z)\right)$, respectively, we obtain

$$\begin{aligned} x_{1,p}(t) = &\cosh\left(\alpha_1^{1/2}\ell_{1,p}^{1/2} t\right) \langle \phi_1(\cdot), \xi_{1,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+ \alpha_1^{1/2}\ell_{1,p}^{-1/2}\sinh\left(\alpha_1^{1/2}\ell_{1,p}^{1/2} t\right) \langle \varphi_1(\cdot), \xi_{1,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+ \int_0^t \alpha_1^{1/2}\ell_{1,p}^{-1/2}\sinh\left(\ell_{1,p}^{1/2}(t-z)\right) \mathcal{R}_{1,p}(t, x_{1,p}(t), x_{2,p}(t))\mathrm{d}z \end{aligned} \tag{14}$$

and

$$\begin{aligned} x_{2,p}(t) = &\cosh\left(\alpha_2^{1/2}\ell_{2,p}^{1/2}t\right)\langle\phi_2(\cdot),\xi_{2,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+\alpha_2^{1/2}\ell_{2,p}^{-1/2}\sinh\left(\alpha_2^{1/2}\ell_{2,p}^{1/2}t\right)\langle\varphi_2(\cdot),\xi_{2,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+\int_0^t \alpha_2^{1/2}\ell_{2,p}^{-1/2}\sinh\left(\ell_{2,p}^{1/2}(t-z)\right)\mathcal{R}_{2,p}(t,x_{1,p}(t),x_{2,p}(t))\mathrm{d}z. \end{aligned} \tag{15}$$

Combining (14), (15) and the interpretation through Fourier series in form

$$x_i(s,t) = \lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}} x_{i,p}(t)\xi_{i,p}(s).$$

We get

$$\begin{aligned} x_1(s,t) = \lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}} \Bigg[&\cosh\left(\alpha_1^{1/2}\ell_{1,p}^{1/2}t\right)\langle\phi_1(\cdot),\xi_{1,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+\alpha_1^{1/2}\ell_{1,p}^{-1/2}\sinh\left(\alpha_1^{1/2}\ell_{1,p}^{1/2}t\right)\langle\varphi_1(\cdot),\xi_{1,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+\int_0^t \alpha_1^{1/2}\ell_{1,p}^{-1/2}\sinh\left(\ell_{1,p}^{1/2}(t-z)\right)\mathcal{R}_{1,p}(t,x_{1,p}(t),x_{2,p}(t))\mathrm{d}z\Bigg]\xi_{1,p}(s) \end{aligned} \tag{16}$$

and

$$\begin{aligned} x_2(s,t) = \lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}} \Bigg[&\cosh\left(\alpha_2^{1/2}\ell_{i,p}^{1/2}t\right)\langle\phi_2(\cdot),\xi_{2,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+\alpha_2^{1/2}\ell_{2,p}^{-1/2}\sinh\left(\alpha_2^{1/2}\ell_{2,p}^{1/2}t\right)\langle\varphi_2(\cdot),\xi_{2,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} \\ &+\int_0^t \alpha_2^{1/2}\ell_{2,p}^{-1/2}\sinh\left(\ell_{2,p}^{1/2}(t-z)\right)\mathcal{R}_{2,p}(t,x_{1,p}(t),x_{2,p}(t))\mathrm{d}z\Bigg]\xi_{2,p}(s). \end{aligned} \tag{17}$$

The proof of Theorem 3.2 is completed. □

3.2. Ill-posedness of the solution of Problem (1)-(3)

As is known, when we observe the solution (41)-(42) that when $p\to\infty$, the rapid escalation of two items exponential in the functions $\sinh(x)$ and $\cosh(x)$ lead to the ill-posedness of Problem (1). For a reason such as a small perturbation in the input data, it shows that a large error in the output data. To demonstrate this, we give an example as follows.

Example. For any $p\in\mathbb{N}$, we choose $\alpha_i=\gamma_i=\delta_i=1$, $\beta_i=0$ for $i\in\{1,2\}$ and the conditions as follow

$$(\phi_1,\varphi_1) := (\phi_1^*,\varphi_1^*) = (0,\ell_{1,p}^{-1/2}\xi_{1,p}), \tag{18}$$

$$(\phi_2,\varphi_2) := (\phi_2^*,\varphi_2^*) = (0,\ell_{2,p}^{-1/2}\xi_{2,p}), \tag{19}$$

$$(F_1,F_2) := (F_1^*,F_2^*) = (0,0). \tag{20}$$

In addition, we choose the following source functions

$$f_i(x_1,x_2) := f_i^*(x_1,x_2) = \lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}} \frac{(\ell_{i,1})^{1/2}\Big(\langle x_1,\xi_{i,p}\rangle + \langle x_2,\xi_{i,p}\rangle\Big)}{3T\exp(T\ell_{i,p}^{1/2})}\xi_{i,p}. \tag{21}$$

Then we prove that the Problem (1) with the input data (18)-(21) is ill-posed problem in the sense of Hadamard.

Solving. To prove that the Problem (1) base on the data (18)-(21) is ill-posed problem i.e., it does not satisfy the third property which were called ill-posed in the sense of Hadamard, we need to do two tasks as follows.

First task. The existence and uniqueness of the solution of Problem (1).
We recall the solution of Problem (1) according to (41)-(42) and the conditions (18)-(21) which is given by

$$x_i(s,t) = \lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}}\Bigg[\ell_{i,p}^{-1/2}\sinh\left(\ell_{i,p}^{1/2}t\right)\langle\varphi_i^*(\cdot),\xi_{i,p}(\cdot)\rangle_{\mathscr{L}^2(\mathcal{D})} + \int_0^t \ell_{i,p}^{-1/2}\sinh\left(\ell_{i,p}^{1/2}(t-z)\right)\left\langle\mathcal{R}_i^*(x_1(\cdot,z),x_2(\cdot,z)),\xi_{i,p}(\cdot,z)\right\rangle_{\mathscr{L}^2(\mathcal{D})}\mathrm{d}z\Bigg]\xi_{i,p}(s). \tag{22}$$

Let $X=(x_1,x_2)\in(\mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D})))^2$ which are satisfy (22). We define the operators $U(X)(t)$ and $V(X)(t)$ as follows

$$U(X)(t) = \lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}}\Bigg[\ell_{1,p}^{-1/2}\sinh\left(\ell_{1,p}^{1/2}t\right)\varphi_{1,p}^*(t) + \int_0^t \ell_{1,p}^{-1/2}\sinh\left(\ell_{1,p}^{1/2}(t-z)\right)f_{1,p}^*(X(z))\mathrm{d}z\Bigg]\xi_{1,p}. \tag{23}$$

$$V(X)(t) = \lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}}\Bigg[\ell_{2,p}^{-1/2}\sinh\left(\ell_{2,p}^{1/2}t\right)\varphi_{2,p}^*(t) + \int_0^t \ell_{2,p}^{-1/2}\sinh\left(\ell_{2,p}^{1/2}(t-z)\right)f_{2,p}^*(X(z))\mathrm{d}z\Bigg]\xi_{2,p}. \tag{24}$$

Then we get

$$\begin{aligned}
&\|U(X)(t)-U(Y)(t)\|_{\mathscr{L}^2(\mathcal{D})}\\
&\le \int_0^t\left\|\lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}}\ell_{1,p}^{-1/2}\sinh\left(\ell_{1,p}^{1/2}(t-z)\right)\left\langle f_1^*(X(z))-f_1^*(Y(z)),\xi_{1,p}\right\rangle_{\mathscr{L}^2(\mathcal{D})}\right\|\mathrm{d}z\\
&\le \int_0^t\left(\lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}}\ell_{1,p}^{-1}\sinh^2\left(\ell_{1,p}^{1/2}(t-z)\right)\left\langle f_1^*(X(z))-f_1^*(Y(z)),\xi_{1,p}\right\rangle^2_{\mathscr{L}^2(\mathcal{D})}\right)^{1/2}\mathrm{d}z.
\end{aligned} \tag{25}$$

Using the inequality $\sinh^2\left(\ell_{1,p}^{1/2}(t-z)\right)\le\exp\left(2\ell_{1,p}^{1/2}(t-z)\right)\le\exp\left(2\ell_{1,p}^{1/2}T\right)$ for $0<t<z<T$, one has

$$\begin{aligned}
&\|U(X)(t)-U(Y)(t)\|_{\mathscr{L}^2(\mathcal{D})}\\
&\le \int_0^t\left(\lim_{\mathcal{P}\to\infty}\sum_{p=1}^{\mathcal{P}}\ell_{1,p}^{-1}\exp\left(2\ell_{1,p}^{1/2}T\right)\left\langle f_1^*(X(z))-f_1^*(Y(z)),\xi_{1,p}\right\rangle^2_{\mathscr{L}^2(\mathcal{D})}\right)^{1/2}\mathrm{d}z.
\end{aligned} \tag{26}$$

Applying (21), we obtain

$$\|U(X)(t)-U(Y)(t)\|_{\mathscr{L}^2(\mathcal{D})}$$

$$
\begin{aligned}
&\leq \int_0^t \left(\lim_{\mathcal{P}\to\infty} \sum_{p=1}^{\mathcal{P}} \ell_{1,p}^{-1} \sinh^2\left(\ell_{1,p}^{1/2}(t-z)\right) \frac{\ell_{1,p}\langle X(z)-Y(z),\xi_{1,p}\rangle^2}{9T^2\exp(2T\ell_{1,p}^{1/2})} \right)^{1/2} \mathrm{d}z \\
&= \int_0^t \frac{\|X(z)-Y(z)\|_{(\mathscr{L}^2(\mathcal{D}))^2}}{3T}\mathrm{d}z \leq \|X-Y\|_{(\mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D})))^2} \int_0^t \frac{1}{3T}\mathrm{d}z \\
&\leq \frac{1}{3}\|X-Y\|_{(\mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D})))^2}, \text{ for all } t\leq T. \qquad (27)
\end{aligned}
$$

By similar approach above, we also obtain

$$
\|V(X)(t)-V(Y)(t)\|_{\mathscr{L}^2(\mathcal{D})} \leq \frac{1}{3}\|X-Y\|_{(\mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D})))^2}, \text{ for all } t\leq T. \qquad (28)
$$

Next, we define an operator $\mathcal{W}(X)(t) = \big(U(X)(t), V(X)(t)\big)$ and combining (27) and (28), we have

$$
\begin{aligned}
\|\mathcal{W}(X)(t)-\mathcal{W}(Y)(t)\|_{(\mathscr{L}^2(\mathcal{D}))^2} &= \|U(X)(t)-U(Y)(t)\|_{\mathscr{L}^2(\mathcal{D})} + \|V(X)(t)-V(Y)(t)\|_{\mathscr{L}^2(\mathcal{D})} \\
&\leq \frac{2}{3}\|X-Y\|_{(\mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D})))^2}. \qquad (29)
\end{aligned}
$$

It leads to estimate as follows

$$
\|\mathcal{W}(X)-\mathcal{W}(Y)\|_{(\mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D})))^2} \leq \frac{2}{3}\|X-Y\|_{(\mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D})))^2}. \qquad (30)
$$

Since the estimation (30), we conclude that $\mathcal{W}$ is a contraction mapping. Applying the Banach fixed point theorem, then we obtain that $\mathcal{W}(X) = X$ has a unique solution $X \in \mathscr{C}([0,T];\mathscr{L}^2(\mathcal{D}))$.

Second task. The instability of the solution (22). By using the inequality $\|a+b\|_{\mathcal{H}} \geq \|a\|_{\mathcal{H}} - \|b\|_{\mathcal{H}}$, we obtain

The instability of the solution of the system (13). By using the inequality $\|c+d\|_{\mathcal{H}} \geq \|c\|_{\mathcal{H}} - \|d\|_{\mathcal{H}}$, we have

$$
\|u_j(t)\|_{\mathcal{H}} \geq \|\mathbf{P}(t)\chi_{1,j}\|_{\mathcal{H}} - \left\| \int_0^t \mathbf{Q}(t-s)\mathcal{F}_0(u_j(s),v_j(s))ds \right\|_{\mathcal{H}}. \qquad (31)
$$

It holds

$$
\begin{aligned}
\left\| \int_0^t \mathbf{Q}(t-s)\mathcal{F}_0(u_j(s),v_j(s))ds \right\|_{\mathcal{H}}^2 &\leq t\int_0^t \left\|\mathbf{Q}(t-s)\mathcal{F}_0(u_j(s),v_j(s))\right\|_{\mathcal{H}}^2 ds \\
&= t\int_0^t \sum_{n=1}^{\infty} \left| \frac{\sinh\left(\sqrt{\lambda_n}(t-s)\right)}{\sqrt{\lambda_n}} \right|^2 \left\langle \mathcal{F}_0(u_j(s),v_j(s)),\Phi_n\right\rangle_{\mathcal{H}}^2 ds \\
&\leq T\int_0^t \sum_{n=1}^{\infty} \frac{\lambda_1 e^{2(t-s-T)\sqrt{\lambda_n}}}{16T^2\lambda_n} \left(\langle u_j(s),\Phi_n\rangle_{\mathcal{H}} + \langle v_j(s),\Phi_n\rangle_{\mathcal{H}} \right)^2 ds \\
&\leq \frac{1}{8T}\int_0^t \left(\|u_j(s)\|_{\mathcal{H}}^2 + \|v_j(s)\|_{\mathcal{H}}^2 \right) ds \\
&\leq \frac{1}{8}\left(\|u_j\|_{C([0,T];\mathcal{H})}^2 + \|v_j\|_{C([0,T];\mathcal{H})}^2 \right). \qquad (32)
\end{aligned}
$$

Combining (31), (32) and using the inequality $2(c-d)^2 \geq c^2 - 2d^2$, for any real numbers c, d, we obtain

$$
2\|u_j(t)\|_{\mathcal{H}}^2 \geq 2\left(\|\mathbf{P}(t)\chi_{1,j}\|_{\mathcal{H}} - \left\| \int_0^t \mathbf{Q}(t-s)\mathcal{F}_0(u_j(s),v_j(s))ds \right\|_{\mathcal{H}} \right)^2
$$

$$\geq \|\mathbf{P}(t)\chi_{1,j}\|^2_{\mathcal{H}} - 2\Big\| \int_0^t \mathbf{Q}(t-s)\mathcal{F}_0(u_j(s), v_j(s))ds \Big\|^2_{\mathcal{H}}$$
$$\geq \frac{\cosh^2\left(\sqrt{\lambda_j}t\right)}{\lambda_j^2} - \frac{1}{4}\Big(\|u_j\|^2_{C([0,T];\mathcal{H})} + \|v_j\|^2_{C([0,T];\mathcal{H})}\Big).$$

This impplies that

$$2\sup_{0\leq t\leq T}\|u_j(t)\|^2_{\mathcal{H}} + \frac{1}{4}\Big(\|u_j\|^2_{C([0,T];\mathcal{H})} + \|v_j\|^2_{C([0,T];\mathcal{H})}\Big) \geq \sup_{0\leq t\leq T}\frac{\cosh^2\left(\sqrt{\lambda_j}t\right)}{\lambda_j^2} \geq \frac{e^{2T\sqrt{\lambda_j}}}{\lambda_j^2}.$$

Hence ,

$$2\|u_j\|^2_{C([0,T];\mathcal{H})} + \frac{1}{4}\Big(\|u_j\|^2_{C([0,T];\mathcal{H})} + \|v_j\|^2_{C([0,T];\mathcal{H})}\Big) \geq \frac{e^{2T\sqrt{\lambda_j}}}{\lambda_j^2}. \tag{33}$$

By a similar way, we also show that

$$2\|v_j\|^2_{C([0,T];\mathcal{H})} + \frac{1}{4}\Big(\|u_j\|^2_{C([0,T];\mathcal{H})} + \|v_j\|^2_{C([0,T];\mathcal{H})}\Big) \geq \frac{e^{2T\sqrt{\widetilde{\lambda}_j}}}{\widetilde{\lambda}_j^2}. \tag{34}$$

This follows from (33) and (34) that

$$\|u_j\|^2_{C([0,T];\mathcal{H})} + \|v_j\|^2_{C([0,T];\mathcal{H})} \geq \frac{2}{5}\left[\frac{e^{2T\sqrt{\lambda_j}}}{\lambda_j^2} + \frac{e^{2T\sqrt{\widetilde{\lambda}_j}}}{\widetilde{\lambda}_j^2}\right].$$

As $j \to +\infty$, we see that

$$\|\chi_{1,j}\|_{\mathcal{H}} + \|\chi_{2,j}\|_{\mathcal{H}} \to 0,\ \|u_j\|_{C([0,T];\mathcal{H})} + \|v_j\|_{C([0,T];\mathcal{H})} \to +\infty.$$

Therefore, the problem (1) does not have a well-defined solution in most cases, according to Hadamard's definition. □

4. Some illustrated numerical examples and discussion

In this section, we consider some examples of illustrating the theory given in section 1. Here, we choose some input data. In order words, we focus on the case one-dimensional space for $d = 1$. At the beginning, let $T = 1$, we recall the problem to find the solution $\mathbf{x} = (x_1, x_2)$ satisfies the nonlinear parabolic equation as follows

$$\begin{cases} \dfrac{\partial^2 x_1}{\partial t^2} = \alpha_1\dfrac{\partial^2 x_1}{\partial s^2} + \beta_1\sinh(\gamma_1 x_1 + \delta_1 x_2) + f_1(x_1, x_2) + F_1(s,t), & (s,t) \in \mathcal{D}\times\mathcal{T}, \\ \dfrac{\partial^2 x_2}{\partial t^2} = \alpha_2\dfrac{\partial^2 x_2}{\partial s^2} + \beta_2\sinh(\gamma_2 x_1 + \delta_2 x_2) + f_2(x_1, x_2) + F_2(s,t), & (s,t) \in \mathcal{D}\times\mathcal{T}, \end{cases} \tag{35}$$

where the domain $\mathcal{D} = (0, \pi)$. Next, we look at some examples for the two cases that we discussed before. We use Python software on a Windows 10 laptop with an Intel(R) Core(TM) i7-11370H CPU@3.30GHz, 8GB RAM and a NVIDIA GeForce RTX 3050Ti GPU with 4GB memory to run the simulations in serial, where CPU times (in seconds) are estimated by Python Calculate Runtime in library `import time` and command code

`start = time.time(), end = time.time().`

Next, we present the results in the following sub-sections.

We test the problem (35) with $(s,t) \in (0,\pi) \times (0,1)$. In $\mathscr{L}^2(0,\pi)$, the orthonormal eigenbasis is giving by $\xi_p = \sqrt{\dfrac{2}{\pi}} \sin(ps)$ and the eigenvalue can be $\ell_p = p^2,\ p = 1,2,...$ Then we have the problem

$$\begin{cases} \dfrac{\partial^2 x_1}{\partial t^2} = \alpha_1 \dfrac{\partial^2 x_1}{\partial s^2} + \beta_1 \sinh(\gamma_1 x_1 + \delta_1 x_2) + f_1(x_1,x_2) + F_1(s,t), & (s,t) \in (0,\pi) \times (0,1), \\ \dfrac{\partial^2 x_2}{\partial t^2} = \alpha_2 \dfrac{\partial^2 x_2}{\partial s^2} + \beta_2 \sinh(\gamma_2 x_1 + \delta_2 x_2) + f_2(x_1,x_2) + F_2(s,t), & (s,t) \in (0,\pi) \times (0,1), \end{cases} \tag{36}$$

and associated the initial condition which is giving by

$$\begin{cases} x_1(s,t) = \phi_1(s) \text{ and } \dfrac{\partial}{\partial t} x_1(s,t) = \varphi_1(s), & (s,t) \in \mathcal{D} \times \{0\}, \\ x_2(s,t) = \phi_2(s) \text{ and } \dfrac{\partial}{\partial t} x_2(s,t) = \varphi_2(s), & (s,t) \in \mathcal{D} \times \{0\}, \end{cases} \tag{37}$$

where we suppose that the measurements are described as follows

$$\phi_1^{\text{obs}} = \phi_1 + \text{"noise"}, \quad \phi_2^{\text{obs}} = \phi_2 + \text{"noise"}, \tag{38}$$

and

$$\varphi_1^{\text{obs}} = \varphi_1 + \text{"noise"}, \quad \varphi_2^{\text{obs}} = \varphi_2 + \text{"noise"}. \tag{39}$$

Next, we choose the input data as follows

$$\begin{cases} f_1(x_1,x_2) = f_2(x_1,x_2) = x_1(s,t)x_2(s,t), & (s,t) \in (0,\pi) \times (0,1), \\ F_1(s,t) = (4-2t)e^{-t^2} \sinh(x_1)\sinh(2x_2), & (s,t) \in (0,\pi) \times (0,1), \\ F_2(s,t) = e^{t^2} \sinh(x_1)\sinh(2x_2), & (s,t) \in (0,\pi) \times (0,1). \end{cases} \tag{40}$$

The behavior of the input functions and their noise are shown in Figure 1, respectively. Here we use $\mathcal{N}(0,1)$ is function `numpy.random.randn`$(\cdot)$ in Python software and the Brownian motion Ψ is presented.

The following regularized solution which is given by

$$\begin{aligned} x_1(s,t) = \lim_{\mathcal{P}\to\infty} \sum_{p=1}^{\mathcal{P}} \Bigg[& \cosh\left(\alpha_1^{1/2} \ell_{1,p}^{1/2} t\right) \langle \phi_1(\cdot), \xi_{1,p}(\cdot) \rangle_{\mathscr{L}^2(\mathcal{D})} \\ & + \alpha_1^{1/2} \ell_{1,p}^{-1/2} \sinh\left(\alpha_1^{1/2} \ell_{1,p}^{1/2} t\right) \langle \varphi_1(\cdot), \xi_{1,p}(\cdot) \rangle_{\mathscr{L}^2(\mathcal{D})} \\ & + \int_0^t \alpha_1^{1/2} \ell_{1,p}^{-1/2} \sinh\left(\ell_{1,p}^{1/2}(t-z)\right) \mathcal{R}_{1,p}(t, x_{1,p}(t), x_{2,p}(t)) \mathrm{d}z \Bigg] \xi_{1,p}(s) \end{aligned} \tag{41}$$

and

$$\begin{aligned} x_2(s,t) = \lim_{\mathcal{P}\to\infty} \sum_{p=1}^{\mathcal{P}} \Bigg[& \cosh\left(\alpha_2^{1/2} \ell_{i,p}^{1/2} t\right) \langle \phi_2(\cdot), \xi_{2,p}(\cdot) \rangle_{\mathscr{L}^2(\mathcal{D})} \\ & + \alpha_2^{1/2} \ell_{2,p}^{-1/2} \sinh\left(\alpha_2^{1/2} \ell_{2,p}^{1/2} t\right) \langle \varphi_2(\cdot), \xi_{2,p}(\cdot) \rangle_{\mathscr{L}^2(\mathcal{D})} \end{aligned}$$

$$+\int_0^t \alpha_2^{1/2}\ell_{2,p}^{-1/2}\sinh\left(\ell_{2,p}^{1/2}(t-z)\right)\mathcal{R}_{2,p}(t,x_{1,p}(t),x_{2,p}(t))\mathrm{d}z\Bigg]\xi_{2,p}(s). \quad (42)$$

We have the matrix form of the regularized solution as follows

$$\widehat{U}=\begin{bmatrix}\widehat{u}^N\left(x_1^{\{1\}},x_2^{\{1\}},t\right) & \widehat{u}^N\left(x_1^{\{1\}},x_2^{\{2\}},t\right) & \cdots & \widehat{u}^N\left(x_1^{\{1\}},x_2^{\{N_{x_2}\}},t\right)\\ \widehat{u}^N\left(x_1^{\{2\}},x_2^{\{1\}},t_1\right) & \widehat{u}^N\left(x_1^{\{2\}},x_2^{\{2\}},t\right) & \cdots & \widehat{u}^N\left(x_1^{\{2\}},x_2^{\{N_{x_2}\}},t\right)\\ \vdots & \vdots & \vdots & \vdots\\ \widehat{u}^N\left(x_1^{\{N_{x_1}\}},x_2^{\{1\}},t\right) & \widehat{u}^N\left(x_1^{\{N_{x_1}\}},x_2^{\{2\}},t\right) & \cdots & \widehat{u}^N\left(x_1^{\{N_{x_1}\}},x_2^{\{N_{x_2}\}},t\right)\end{bmatrix}_{N_{x_1}\times N_{x_2}},$$

where N_{x_1} and N_{x_2} are the partition of spaces.

We present the absolute root mean square errors ($\mathbf{err_{abs}}$) between the exact and approximate solutions as follow

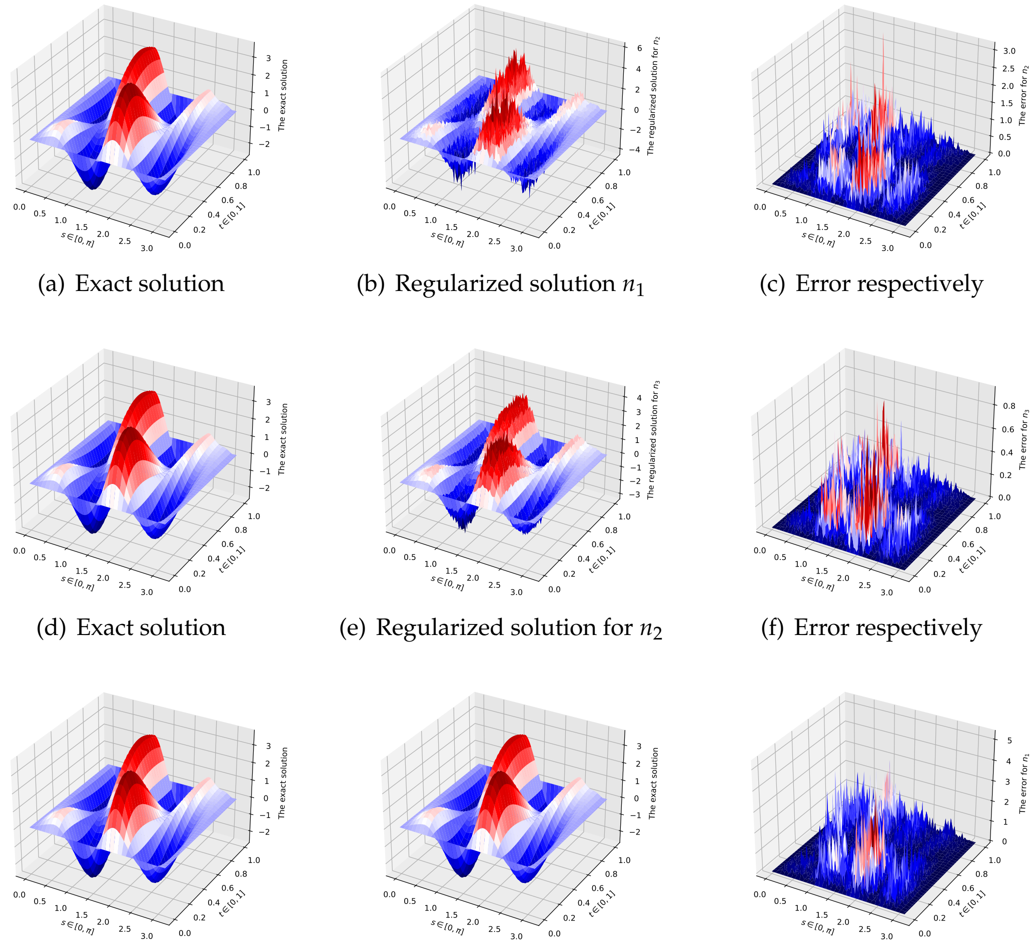
$$\mathbf{err_{abs}}(t)=\left[\frac{1}{n_1n_2}\sum_{i=\overline{1,n_1}}\sum_{j=\overline{1,n_2}}\left(\widehat{u}^N\left(x_1^{\{i\}},x_2^{\{j\}},t\right)-u\left(x_1^{\{i\}},x_2^{\{j\}},t\right)\right)^2\right]^{1/2}.$$

Let the N is a cut-off constants, equal to the greatest integer number less than or equal to $\dfrac{\alpha_0}{T(2\alpha_0+d/2)}\log(n_1n_2)$ for $n_1,n_2\in\{10^3,10^5,10^7\}$

$\mathbf{err_{abs}}$	$n_1=n_2=10^3, N=2$	$n_1=n_2=10^5, N=3$	$n_1=n_2=10^7, N=5$
	Time CPU (seconds): 320	Time CPU (seconds): 455	Time CPU (seconds): 607
$\mathbf{err_{abs}}(0.1)$	2.34665383e-01	1.84868768e-01	5.07467431e-02
$\mathbf{err_{abs}}(0.3)$	4.58432336e-01	1.46894333e-01	8.35688997e-02
$\mathbf{err_{abs}}(0.5)$	3.70542990e-01	3.13578545e-01	2.56785300e-01
$\mathbf{err_{abs}}(0.7)$	3.45661075e-01	2.64328299e-01	1.02457006e-01
$\mathbf{err_{abs}}(0.9)$	7.67975080e-01	5.39633675e-01	3.47537672e-02

TABLE 1. The absolute root mean square errors between the exact and approximate solutions on 2D space case

The results of this section are presented in Tables and Figures. For detail, in 1D case, the input data are given by Figure 1, the graphs of solutions are given in Figure 1 at $t\in\{0.1,0.5,0.9\}$ and Figure 1 for 3D-graph of the solution. Furthermore, to make it easier to compare the exact and regular solutions, we present the error in Table 1. Similar simulation for 2D space case in Figure 1 and Table 1. From these observations, we conclude that the larger n mostly leads to the better approximation. From the observed data above, we can conclude that the proposed method is effective and stable. To make a easy comparison, we show the exact and regularized solutions in Figure 1.



(a) Exact solution (b) Regularized solution n_1 (c) Error respectively

(d) Exact solution (e) Regularized solution for n_2 (f) Error respectively

(g) Exact solution (h) Regularized solution for n_3 (i) Error respectively

FIGURE 1. Graphs of the solutions and error at $t = 0.1$ for n_1, n_2 and n_3 on 2D case

Authors Contributions

All authors read and approved the final version of the manuscript.

Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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