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On nonlinear Sobolev equations with terminal observations in L^p spaces.

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Abstract. In this paper, we investigate the backward problem for the heat equation equipped with the time fractional conformable derivative. This problem is a generalization of the classical heat equation. We consider the problem with a nonlinear source function in a bounded domain. This problem is shown to be ill-posed, so we regularize the solution by the Fourier truncated method and we estimate the error term in the $L^p(\mathcal{D})$ space. An example to illustrate the theory is given in the final section.

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1. Introduction

Various derivatives arise in problems in engineering, economics, physics, etc., for example in the study of the anomalous diffusion process in physics [1, 2], problems related to heat propagation [3, 4], problems in biological engineering [5, 6], problems with partial differential equations in image processing [7], signal processing [8], control theory [9, 10], and uncertainty models with random walks [11,12]; see also [13–19,23,31,33–37] and the references therein. The time fractional conformable derivative [23–25] with order $\alpha \in (0,1)$, denoted by $\frac{\mathscr{C}\partial^\alpha}{\partial t^\alpha}$, is defined as:

$$\frac{\mathscr{C}\partial^\alpha}{\partial t^\alpha} f(t) = \lim_{\delta \to 0} \frac{f\left(t + \delta t^{1-\alpha}\right) - f(t)}{\delta}, \tag{1}$$

for all $t > 0$. If we consider $(0, t_0)$, $t_0 > 0$ and $\lim_{t \to t_0^+} \frac{\mathscr{C}\partial^\alpha}{\partial t^\alpha} f(t)$ exists, then $\frac{\mathscr{C}\partial^\alpha}{\partial t^\alpha} f(t_0) = \lim_{t \to t_0^+} \frac{\mathscr{C}\partial^\alpha}{\partial t^\alpha} f(t)$.

This type of derivative has applications in quantum mechanics [26], the authors in [27] used the conformable Fourier series to express the solution of the conformable fractional heat equation, problems in Newtonian mechanics were investigated with a conformable derivative in [29], and in [28] the authors obtained the stochastic solution by running the processes corresponding to conformable Cauchy problems with nonlinear deterministic clocks; for more applications of the conformable derivative we refer the reader to [42]– [43].

There are also many papers on partial differential equations with different types of derivatives [20–22]. In this paper we will consider the conformable derivative and the heat equation, namely:

$$\begin{cases} \frac{^{\mathscr{C}}\partial^{\alpha}}{\partial t^{\alpha}} w(x,t) + \big(-\Delta\big)^{s} w(x,t) - b\frac{^{\mathscr{C}}\partial^{\alpha}}{\partial t^{\alpha}} \Delta w(x,t) = F(w(x,t)), & x \in \mathcal{D},\ \ t \in (0,T), \\ w(x,t) = 0, & x \in \partial\mathcal{D}, t \in (0,T), \\ w(x,T) = g(x), & x \in \mathcal{D}. \end{cases} \tag{2}$$

Here $\mathcal{D} \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with the smooth boundary $\partial\mathcal{D}$, and $T > 0$.

In this work, we focus on studying the well-posedness and the regularization process. In Hadamard's sense, a problem is well-posed if it satisfies three conditions, where the third condition is "continuous dependence of the solution on the given data". For partial differential equation problems, this third condition is frequently not satisfied which results in an ill-posed problem. Therefore, in order for the obtained solution to depend continuously on the observed data, we need to modify the problem to a class of other well-posed problems. This technique is called the regularization method and we refer the reader to [32, 38–41] and the references therein. The regularization method mentioned above all estimate the convergence rate between the exact solution and the regularized solution based on the principle of choosing a priori or a posteriori parameter. To our best of our knowledge, most of the previous studies evaluated the error term mainly in the $L^2(\mathcal{D})$ space. In this article, we obtain results for regularized problems in $L^q(\mathcal{D})$ for $q \neq 2$. The evaluation in $L^q(\mathcal{D})$ spaces for $q \neq 2$ is quite difficult and to achieve this we use some embeddings between $L^q(\mathcal{D})$ and the Hilbert scales space $\mathbb{X}^r(\mathcal{D})$.

The paper is organized as follows. Section 2 gives some preliminaries to be used later. In section 3, we derive the formula of the mild solution to (2), and show that (2) is ill-posed in Hadamard's sense. Section 4 considers the error estimation and solution regularization. The last section presents some numerical examples to demonstrate the obtained results.

2. Preliminary results

Definition 2.1. Let us consider the operator $\big(-\Delta\big)^s$ on $\mathcal{H}^1_0(\Omega) \cap \mathcal{H}^2(\Omega)$, and assume that the operator $\big(-\Delta\big)^s$ has the eigenvalues λ_n such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots$ which approach ∞ as n goes to ∞, and $e_n(x)$ is the orthonormal basis in $L^2(\mathcal{D})$. For all $s \geq 0$, we define by $(-\Delta)^s$ the operator

$$\big(-\Delta\big)^s v := \sum_{n=1}^{\infty} \langle v, e_n \rangle \lambda_n^s e_n.$$

The Hilbert scale space is

$$\mathbb{X}^r(\mathcal{D}) = \Big\{ f \in L^2(\mathcal{D}),\ \sum_{n=1}^{\infty} \lambda_n^{2r} \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big)^2 < \infty \Big\},$$

for any $r \geq 0$. Now $\mathbb{X}^r(\mathcal{D})$ is a Hilbert space with norm:

$$\|f\|_{\mathbb{X}^r(\mathcal{D})} = \Bigg(\sum_{n=1}^{\infty} \lambda_n^{2r} \Big(\int_{\mathcal{D}} f(x) e_n(x) dx \Big)^2 \Bigg)^{1/2},\ f \in \mathbb{X}^r(\mathcal{D}). \tag{3}$$

Lemma 2.2. (See [30]) The following are true:

$$\left.\begin{aligned} L^q(\mathcal{D}) &\hookrightarrow \mathbb{X}^\mu(\mathcal{D}), \quad \text{if} \quad -\frac{N}{4} < \mu \le 0, \quad q \ge \frac{2N}{N-4\mu}, \\ \mathbb{X}^\mu(\mathcal{D}) &\hookrightarrow L^q(\mathcal{D}), \quad \text{if} \quad 0 \le \mu < \frac{N}{4}, \quad q \le \frac{2N}{N-4\mu}. \end{aligned}\right\} \tag{4}$$

3. Inverse initial problem (2)

The mild solution of (2) is obtained by the Fourier series $w(x,t) = \sum_{n=1}^{\infty} w_n(t)e_n(x)$, with $w_n(t) = \int_{\mathcal{D}} w(x,t)e_n(x)dx$. We have

$$\begin{cases} \dfrac{{}^{\mathscr{C}}\partial^\alpha}{\partial t^\alpha}\langle w(.,t),e_n\rangle + b\lambda_n \dfrac{{}^{\mathscr{C}}\partial^\alpha}{\partial t^\alpha}\langle w(.,t),e_n\rangle - \lambda_n^s\langle w(.,t),e_n\rangle &= \langle F(\cdot,t),e_n\rangle, \quad t\in(0,T), \\ \langle w(.,0),e_n\rangle = \langle w_0,e_n\rangle. \end{cases} \tag{5}$$

Thus

$$\frac{{}^{\mathscr{C}}\partial^\alpha}{\partial t^\alpha} w_n(t) - \lambda_n^s(1+b\lambda_n)^{-1}w_n(t) = (1+b\lambda_n)^{-1}F_n(t), \text{ where } F_n(t) = \int_{\mathcal{D}} F(x,t)e_n(x)dx. \tag{6}$$

Based on [31], we have

$$\begin{aligned} \langle w(.,t),e_n\rangle &= \exp\big(-\lambda_n^s(1+b\lambda_n)^{-1}t^\alpha\alpha^{-1}\big)\Big(\int_{\mathcal{D}} w_0(x)e_n(x)dx\Big) \\ &\quad + \left(1+b\lambda_n\right)^{-1}\int_0^t \mu^{\alpha-1}\exp\left(\lambda_n^s(1+b\lambda_n)^{-1}\left(\mu^\alpha - t^\alpha\right)\alpha^{-1}\right)F_n(\mu)d\mu. \end{aligned} \tag{7}$$

Letting $t = T$, it follows from (7) that

$$\begin{aligned} \Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big) &= \exp\Big(-\lambda_n^s(1+b\lambda_n)^{-1}T^\alpha\alpha^{-1}\Big)\Big(\int_{\mathcal{D}} w_0(x)e_n(x)dx\Big) \\ &\quad + (1+b\lambda_n)^{-1}\int_0^T \mu^{\alpha-1}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(\mu^\alpha - t^\alpha)\alpha^{-1}\Big)F_n(\mu)d\mu. \end{aligned} \tag{8}$$

From (8), we get

$$\begin{aligned} \Big(\int_{\mathcal{D}} w_0(x)e_n(x)dx\Big) &= \Big(\exp\Big(-\lambda_n^s\left(1+b\lambda_n\right)^{-1}T^\alpha\alpha^{-1}\Big)\Big)^{-1}\Big[\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big) \\ &\quad - (1+b\lambda_n)^{-1}\int_0^T \mu^{\alpha-1}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(\mu^\alpha - t^\alpha)\alpha^{-1}\Big)F_n(\mu)d\mu\Big]. \end{aligned} \tag{9}$$

Substituting (9) into (8) and we obtain

$$\begin{aligned} \langle w(.,t),e_n\rangle &= \exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(T^\alpha - t^\alpha)\alpha^{-1}\Big)\Big(\int_{\mathcal{D}} g(x)e_n(x)dx\Big) \\ &\quad - (1+b\lambda_n)^{-1}\int_t^T \mu^{\alpha-1}\exp\Big(\lambda_n^s\left(1+b\lambda_n\right)^{-1}(\mu^\alpha - t^\alpha)\alpha^{-1}\Big)F_n(\mu)d\mu. \end{aligned} \tag{10}$$

This leads to

$$w(x,t)=\sum_{n=1}^{+\infty}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\Big)\Big(\int_{\mathcal{D}}g(x)e_n(x)dx\Big)e_n(x)$$
$$-\sum_{n=1}^{+\infty}(1+b\lambda_n)^{-1}\Big(\int_t^T\mu^{\alpha-1}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\Big)F_n(\mu)d\mu\Big)e_n(x). \quad (11)$$

From (11), we have

$$w(x,t)=\sum_{n=1}^{+\infty}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\Big)\Big(\int_{\mathcal{D}}g(x)e_n(x)dx\Big)e_n(x)$$
$$-\sum_{n=1}^{+\infty}(1+b\lambda_n)^{-1}\Big(\int_t^T\mu^{\alpha-1}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\Big)F_n(w(\mu))d\mu\Big)e_n(x). \quad (12)$$

3.1. The ill-posedness of problem (2)

For any $k\in\mathbb{N}^*$, where N^* is the set of non-negative natural numbers. Let g and $F(w)$ be

$$\begin{cases} g(x):=\lambda_k^{-1/2}e_k(x),\\ F(w):=\sum_{k=1}^{\infty}\dfrac{\alpha(1+b\lambda_k)}{2\mathcal{C}_F T^\alpha\exp\Big(\frac{\lambda_k^s}{1+b\lambda_k}\frac{T^\alpha}{\alpha}\Big)}\Big(\int_{\mathcal{D}}w(x)e_k(x)dx\Big)e_k(x).\end{cases} \quad (13)$$

let w satisfy the integral equation

$$w(x,t)=\sum_{n=1}^{+\infty}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\Big)\Big(\int_{\mathcal{D}}g(x)e_n(x)dx\Big)e_n(x)$$
$$-\sum_{n=1}^{+\infty}(1+b\lambda_n)^{-1}\Big(\int_t^T\mu^{\alpha-1}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\Big)F_n(w(\mu))d\mu\Big)e_n(x). \quad (14)$$

For $w\in L^\infty(0,T;L^2(\mathcal{D}))$, let

$$\mathcal{R}(w)(x,t)=\sum_{n=1}^{+\infty}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\Big)\Big(\int_{\mathcal{D}}g(x)e_n(x)dx\Big)e_n(x)$$
$$-\sum_{n=1}^{+\infty}(1+b\lambda_n)^{-1}\Big(\int_t^T\mu^{\alpha-1}\exp\Big(\lambda_n^s(1+b\lambda_n)^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\Big)F_n(w(\mu))d\mu\Big)e_n(x).$$

Then for any $w_1,w_2\in L^\infty(0,T;L^2(\mathcal{D}))$, using Hölder's inequality, we obtain

$$\|\mathcal{R}(w_1)(\cdot,t)-\mathcal{R}(w_2)(\cdot,t)\|^2_{L^2(\mathcal{D})}$$
$$=\sum_{n=1}^{\infty}\frac{1}{(1+b\lambda_n)^2}\Big|\int_t^T\mu^{\alpha-1}\exp\Big(\lambda_n^{s-1}b^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\Big)$$
$$\times\Big(\int_{\mathcal{D}}(F(w_1)(\mu)-F(w_2)(\mu))e_n(x)dx\Big)d\mu\Big|^2$$
$$\le\frac{\alpha}{4T^\alpha}\sum_{n=1}^{\infty}\Bigg(\int_t^T\mu^{\alpha-1}d\mu\int_t^T\mu^{\alpha-1}\exp\big(2\lambda_n^{s-1}b^{-1}(\mu^\alpha-t^\alpha-T^\alpha)\alpha^{-1}\big)$$

$$\times \left(\int_{\mathcal{D}} ((w_1)(\mu) - (w_2)(\mu)) e_n(x) dx \right) d\mu \Big)$$

$$\leq \frac{\alpha}{4T^\alpha} \int_t^T \mu^{\alpha-1} \|w_1(\cdot,\mu) - w_2(\cdot,\mu)\|^2_{L^2(\mathcal{D})} d\mu \leq 4^{-1} \|w_1 - w_2\|^2_{L^\infty(0,T;L^2(\mathcal{D}))} . \tag{15}$$

This implies that

$$\|\mathcal{R}(w_1) - \mathcal{R}(w_2)\|_{L^\infty(0,T;L^2(\mathcal{D}))} \leq 2^{-1} \|w_1 - w_2\|_{L^\infty(0,T;L^2(\mathcal{D}))} .$$

Hence $\mathcal{R}$ is a contraction. We conclude that $\mathcal{R}(w) = w$ has a unique solution $w \in L^\infty(0,T;L^2(\mathcal{D}))$ from Banach's contraction theorem. From (14), we observe that

$$\|w(\cdot,t)\|_{L^2(\mathcal{D})} \geq \Big\| \underbrace{\sum_{n=1}^{\infty} \Big(\exp\left(\lambda_n^s (1+b\lambda_n)^{-1} (T^\alpha - t^\alpha)\alpha^{-1}\right) \int_{\mathcal{D}} g(x) e_n(x) dx \Big) e_n(x)}_{\mathcal{R}_1(x,t)} \Big\|_{L^2(\mathcal{D})}$$

$$- \Big\| \underbrace{\sum_{n=1}^{\infty} (1+b\lambda_n)^{-1} \left(\int_t^T \mu^{\alpha-1} \exp\left(\lambda_n^s (1+b\lambda_n)^{-1} (\mu^\alpha - t^\alpha)\alpha^{-1}\right) \left(\int_{\mathcal{D}} F(x,\mu,w(x,\mu)) e_n(x) dx \right) d\mu \right) e_n(x)}_{\mathcal{R}_2(x,t)} \Big\|_{L^2(\mathcal{D})} . \tag{16}$$

From $g(x) = \lambda_k^{-\frac{1}{2}} e_k(x)$ for any $k \in \mathbb{N}^*$ and note that $\|e_k\|_{L^2(\mathcal{D})} = 1$, we have

$$\begin{aligned} \|\mathcal{R}_1(\cdot,t)\|_{L^2(\mathcal{D})} &= \left(\sum_{n=1}^{\infty} \left| \exp\left(\lambda_n^s (1+b\lambda_n)^{-1} (T^\alpha - t^\alpha)\alpha^{-1}\right) \left(\int_{\mathcal{D}} g(x) e_n(x) dx \right) \right|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{n=1}^{\infty} \left| \exp\left(\lambda_n^s (1+b\lambda_n)^{-1} (T^\alpha - t^\alpha)\alpha^{-1}\right) \left(\int_{\mathcal{D}} \lambda_k^{-\frac{1}{2}} e_k(x) e_n(x) dx \right) \right|^2 \right)^{\frac{1}{2}} \\ &\geq \lambda_k^{-\frac{1}{2}} \exp\left(\lambda_k^s (1+b\lambda_k)^{-1} (T^\alpha - t^\alpha)\alpha^{-1}\right) \|e_k\|_{L^2(\mathcal{D})} \\ &= \lambda_k^{-\frac{1}{2}} \exp\left(\lambda_k^s (1+b\lambda_k)^{-1} (T^\alpha - t^\alpha)\alpha^{-1}\right) . \end{aligned} \tag{17}$$

By an argument similar to (15), we have

$$\|\mathcal{R}_2 u(\cdot,t)\|_{L^2(\mathcal{D})} \leq 2^{-1} \|u\|_{L^\infty(0,T;L^2(\mathcal{D}))} . \tag{18}$$

Combining (16)-(18), yields

$$\|u(\cdot,t)\|_{L^2(\mathcal{D})} \geq \lambda_k^{-\frac{1}{2}} \exp\left(\lambda_k^s (1+b\lambda_k)^{-1} (T^\alpha - t^\alpha)\alpha^{-1}\right) - \frac{1}{2} \|u\|_{L^\infty(0,T;L^2(\mathcal{D}))} ,$$

which implies that

$$\|u\|_{L^\infty(0,T;L^2(\mathcal{D}))} \geq \sup_{t\in[0,T]} \frac{2}{3} \lambda_k^{-\frac{1}{2}} \exp\left(\lambda_k^s (1+b\lambda_k)^{-1} (T^\alpha - t^\alpha)\alpha^{-1} (T^\alpha - t^\alpha)\alpha^{-1}\right) . \tag{19}$$

We also have

$$\sup_{t\in[0,T]} \exp\left(\lambda_k^s (1+b\lambda_k)^{-1} (T^\alpha - t^\alpha)\alpha^{-1}\right) = \exp\left(\lambda_k^{s-1} b^{-1} T^\alpha \alpha^{-1}\right) . \tag{20}$$

Consequently, (19) becomes

$$\|u(\cdot,.)\|_{L^\infty(0,T;L^2(\mathcal{D}))} \geq \frac{2}{3}\,\lambda_k^{-\frac{1}{2}}\,\exp\left(\lambda_k^{s-1}b^{-1}T^\alpha\alpha^{-1}\right). \tag{21}$$

Then, by letting $k\to\infty$, one has

$$\lim_{k\to\infty}\|g\|_{L^2(\mathcal{D})} = \lim_{k\to\infty}\lambda_k^{-\frac{1}{2}} = 0,$$

but

$$\lim_{k\to\infty}\|u\|_{L^\infty(0,T;L^2(\mathcal{D}))} \geq \frac{2}{3}\lim_{k\to\infty}\,\lambda_k^{-\frac{1}{2}}\,\exp\left(\lambda_k^{s-1}b^{-1}T^\alpha\alpha^{-1}\right) = \infty.$$

Therefore, problem (2) is ill-posed in the Hadamard sense in $L^\infty\left(0,T;L^2(\mathcal{D})\right)$.

4. Regularization of the nonlinear pseudo-parabolic equation on $L^q(\mathcal{D})$ spaces

For any $\mathcal{N}_{tr}>0$, a regularized solution $\widetilde{U}_\epsilon^{\mathcal{N}_{tr}}(x,t)$ is:

$$\widetilde{U}_\epsilon^{\mathcal{N}_{tr}}(x,t) = \sum_{n=1}^{\mathcal{N}_{tr}}\exp\left(\lambda_n^s(1+b\lambda_n)^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\right)\Big(\int_{\mathcal{D}} g_\epsilon(x)e_n(x)dx\Big)e_n(x)$$
$$-\sum_{n=1}^{\mathcal{N}_{tr}}(1+b\lambda_n)^{-1}\left(\int_t^T\mu^{\alpha-1}\exp\left(\lambda_n^s(1+b\lambda_n)^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\right)F_n(\widetilde{W}_\varepsilon^{\mathcal{N}_{tr}}(\cdot,\mu))d\mu\right)e_n(x). \tag{22}$$

Theorem 4.1. *Assume that $u\in L^\infty\left(0,T;\mathbb{X}^{a+\eta}(\mathcal{D})\right)$ for any $a>0$ and $0<\eta<\frac{N}{4}$, and there exists a constant $\mathcal{M}>0$ such that*

$$\mathcal{M} = \operatorname*{ess\,sup}_{0\leq t\leq T}\left(\sum_{n=1}^\infty\lambda_n^{2\eta_0}\exp\left(2t^\alpha\alpha^{-1}\lambda_n^{s-1}b^{-1}\right)\left|\langle u(t),e_n\rangle\right|^2\right)^{\frac{1}{2}}, \tag{23}$$

where $\eta_0>\eta$, the noisy data $g_\epsilon\in L^q(\mathcal{D})$ is such that

$$\left\|g_\epsilon-g\right\|_{L^q(\mathcal{D})}\leq\epsilon,\quad 1<q<2, \tag{24}$$

and $\mathcal{N}_{tr}$ is chosen such that for any $t>0$

$$\lim_{\epsilon\to0}\mathcal{N}_{tr}=\infty,\ \lim_{\epsilon\to0}(\mathcal{N}_{tr})^{\eta-\frac{Nq-2N}{4q}}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}T^\alpha\alpha^{-1}\right)\epsilon=0. \tag{25}$$

Then $\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{L^{\frac{2N}{N-4\eta}}(\mathcal{D})}$ is of order

$$\max\left\{(\mathcal{N}_{tr})^{-a},(\mathcal{N}_{tr})^{\eta-\eta_0},(\mathcal{N}_{tr})^{\eta-\frac{Nq-2N}{4q}}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}T^\alpha\alpha^{-1}\right)\epsilon\right\}. \tag{26}$$

Remark 4.1. By choosing $\mathcal{N}_{tr}=\left(\dfrac{b(1-\beta)}{\alpha^{-1}T^\alpha}\right)^{\frac{1}{s-1}}\left[\log(\epsilon^{-1})\right]^{\frac{1}{s-1}}$, for $\beta\in(0,1)$, this implies that $\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{L^{\frac{2N}{N-4\eta}}(\mathcal{D})}$ is of order

$$\max\left\{\left[\frac{b(1-\beta)}{\alpha^{-1}T^\alpha}\log(\varepsilon^{-1})\right]^{-\frac{a}{s-1}},\left[\frac{b(1-\beta)}{\alpha^{-1}T^\alpha}\log(\varepsilon^{-1})\right]^{\frac{\eta-\eta_0}{s-1}},\left[\frac{b(1-\beta)}{\alpha^{-1}T^\alpha}\log(\varepsilon^{-1})\right]^{\eta-\frac{Nq-2N}{4q}}\epsilon^\beta\right\}.$$

Proof. The triangle inequality gives

$$\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{L^2(\mathcal{D})} \le \left\|\widetilde{U}_\epsilon(\cdot,t)-\mathbb{T}_{\mathcal{N}_{tr}}u(\cdot,t)\right\|_{L^2(\mathcal{D})}+\left\|u(\cdot,t)-\mathbb{T}_{\mathcal{N}_{tr}}u(\cdot,t)\right\|_{L^2(\mathcal{D})}. \tag{27}$$

From (12), and let

$$\mathcal{P}_1(T-t)f=\sum_{n=1}^{\infty}\exp\left(\lambda_n^s(1+b\lambda_n)^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\right)\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)e_n(x).$$

$$\mathcal{P}_2(T-t)f=\sum_{n=1}^{\infty}(1+b\lambda_n)^{-1}\exp\left(\lambda_n^s(1+b\lambda_n)^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\right)\Big(\int_{\mathcal{D}} f(x)e_n(x)dx\Big)e_n(x). \tag{28}$$

We can rewrite (22) as:

$$\mathbb{T}_{\mathcal{N}_{tr}}u(t)=\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_{tr}}g-\int_t^T \mu^{\alpha-1}\mathcal{P}_2(\mu-t)\mathbb{T}_{\mathcal{N}_{tr}}F(u(\mu))d\mu. \tag{29}$$

From (29), we have

$$\begin{aligned}\widetilde{U}_\varepsilon(t)-\mathbb{T}_{\mathcal{N}_{tr}}u(t)=&\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_{tr}}\left(g_\epsilon-g\right)\\&-\int_t^T \mu^{\alpha-1}\mathcal{P}_2(\mu-t)\left[\mathbb{T}_{\mathcal{N}_{tr}}F(\widetilde{U}_\epsilon(\mu))-\mathbb{T}_{\mathcal{N}_{tr}}F(u(\mu))\right]d\mu.\end{aligned} \tag{30}$$

This implies that

$$\begin{aligned}\left\|\widetilde{U}_\varepsilon(t)-u(t)\right\|_{L^2(\mathcal{D})} &\le \|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_{tr}}\left(g_\varepsilon-g\right)\|_{L^2(\mathcal{D})}+\|u(\cdot,t)-\mathbb{T}_{\mathcal{N}_{tr}}u(\cdot,t)\|_{L^2(\mathcal{D})}\\&\quad+\left\|\int_t^T \mu^{\alpha-1}\mathcal{P}_2(\mu-t)\left[\mathbb{T}_{\mathcal{N}_{tr}}F(\widetilde{U}_\varepsilon(\mu))-\mathbb{T}_{\mathcal{N}_{tr}}F(u(\mu))\right]d\mu\right\|_{L^2(\mathcal{D})}\\&=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.\end{aligned} \tag{31}$$

For $1<q<2$, using the Sobolev embedding in Lemma 2.2, we get

$$\begin{aligned}\mathcal{I}_1&=\left\|\mathcal{P}_1(T-t)\mathbb{T}_{\mathcal{N}_{tr}}\left(g_\epsilon-g\right)\right\|_{L^2(\mathcal{D})}\\&\le(\mathcal{N}_{tr})^{-\frac{Nq-2N}{4q}}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\right)\left\|g_\epsilon-g\right\|_{\mathbb{X}^{\frac{Nq-2N}{4q}}(\mathcal{D})}\\&\le\mathcal{C}_1(N,\eta)\mathcal{C}_2(N,q)(\mathcal{N}_{tr})^{-\frac{Nq-2N}{4q}}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\right)\|g_\epsilon-g\|_{L^q(\mathcal{D})}.\end{aligned} \tag{32}$$

Next,

$$\begin{aligned}\mathcal{I}_2&=\left\|u(t)-\mathbb{T}_{\mathcal{N}_{tr}}u(t)\right\|_{L^2(\mathcal{D})}\\&=\left(\sum_{\lambda_n>\mathcal{N}_{tr}}\exp\left(-2t^\alpha\alpha^{-1}\lambda_n^{s-1}b^{-1}\right)\lambda_n^{-2\eta_0}\exp\left(2t^\alpha\alpha^{-1}\lambda_n^{s-1}b^{-1}\right)\lambda_n^{2\eta_0}\left|\langle u(t),e_j\rangle\right|^2\right)^{\frac12}\\&\le(\mathcal{N}_{tr})^{-\eta_0}\exp\left(-t^\alpha\alpha^{-1}(\mathcal{N}_{tr})^{s-1}b^{-1}\right)\mathcal{M},\end{aligned} \tag{33}$$

using $\lambda_n^{-2\eta_0}\le(\mathcal{N}_{tr})^{-2\eta_0}$ for $\lambda_n>\mathcal{N}_{tr}$ and recalling

$$\mathcal{M}=\operatorname{esssup}_{0\le t\le T}\left(\sum_{n=1}^{\infty}\lambda_n^{2\eta_0}\exp\left(2t^\alpha\alpha^{-1}\lambda_n^{s-1}b^{-1}\right)\left|\langle u(t),e_n\rangle\right|^2\right)^{\frac12}. \tag{34}$$

A similar argument yields

$$
\begin{aligned}
\mathcal{I}_3 &= \left\|\mathcal{P}_2(\mu-t)\mathbb{T}_{\mathcal{N}_{tr}}(F(\widetilde{U}_\epsilon(\mu))-F(u(\mu)))\right\|_{\mathbb{X}^0(\mathcal{D})} \\
&\le (1+b\mathcal{N}_{tr})^{-1}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\right)\left\|F(\widetilde{U}_\epsilon(\mu))-F(u(\mu))\right\|_{L^2(\mathcal{D})} \\
&\le (1+b\mathcal{N}_{tr})^{-1}\,L_F\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\right)\left\|\widetilde{U}_\epsilon(\mu)-u(\mu)\right\|_{L^2(\mathcal{D})} \\
&\le (1+b\mathcal{N}_{tr})^{-1}\,\mathcal{C}_3(N,\eta)L_F\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\right)\left\|\widetilde{U}_\epsilon(\mu)-u(\mu)\right\|_{L^2(\mathcal{D})}.
\end{aligned} \tag{35}
$$

We also find that

$$
\begin{aligned}
\mathcal{I}_3 \le{}& (1+b\mathcal{N}_{tr})^{-1}\,\mathcal{C}_3(N,\eta)L_F \\
&\times\int_t^T \mu^{\alpha-1}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1}\right)\left\|\widetilde{U}_\epsilon(\mu)-u(\mu)\right\|_{L^2(\mathcal{D})}d\mu.
\end{aligned} \tag{36}
$$

Combining (31), (32), (33), and (36), we get

$$
\begin{aligned}
&\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{L^2(\mathcal{D})} \le \left\|\widetilde{U}_\epsilon(\cdot,t)-\mathbb{T}_{\mathcal{N}_{tr}}u(\cdot,t)\right\|_{L^2(\mathcal{D})} + \|u(\cdot,t)-\mathbb{T}_{\mathcal{N}_{tr}}u(\cdot,t)\|_{L^2(\mathcal{D})} \\
&\le \mathcal{C}_1(N,\eta)\mathcal{C}_2(N,q)(\mathcal{N}_{tr})^{-\frac{Nq-2N}{4q}}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}(T^\alpha-t^\alpha)\alpha^{-1}\right)\epsilon \\
&+(\mathcal{N}_{tr})^{-\eta_0}\exp(-t^\alpha\alpha^{-1}(\mathcal{N}_{tr})^{s-1}b^{-1})\mathcal{M} \\
&+(1+b\mathcal{N}_{tr})^{-1}\mathcal{C}_3(N,\eta)L_F\int_t^T\mu^{\alpha-1}\exp((\mathcal{N}_{tr})^{s-1}b^{-1}(\mu^\alpha-t^\alpha)\alpha^{-1})\left\|\widetilde{U}_\epsilon(\mu)-u(\mu)\right\|_{L^2(\mathcal{D})}d\mu.
\end{aligned}
$$

From Gronwall's inequality, multiplying by $e^{\frac{t^\alpha}{\alpha}(\mathcal{N}_{tr})^{s-1}b^{-1}}$ on both sides, we get

$$
\begin{aligned}
&e^{\frac{t^\alpha}{\alpha}(\mathcal{N}_{tr})^{s-1}b^{-1}}\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{L^2(\mathcal{D})} \\
&\le \mathcal{C}_1(N,\eta)\mathcal{C}_2(N,q)(\mathcal{N}_{tr})^{-\frac{Nq-2N}{4q}}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}T^\alpha\alpha^{-1}\right)\epsilon+(\mathcal{N}_{tr})^{-\eta_0}\,\mathcal{M} \\
&+(1+b\mathcal{N}_{tr})^{-1}\,\mathcal{C}_3(N,\eta)L_F\int_t^T\mu^{\alpha-1}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}\mu^\alpha\alpha^{-1}\right)\left\|\widetilde{U}_\epsilon(\mu)-u(\mu)\right\|_{L^2(\mathcal{D})}d\mu.
\end{aligned} \tag{37}
$$

Thus

$$
\begin{aligned}
\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{L^2(\mathcal{D})} &\le e^{-\frac{t^\alpha}{\alpha}(\mathcal{N}_{tr})^{s-1}b^{-1}}\;e^{(1+b\mathcal{N}_{tr})^{-1}(\mathcal{C}_3(N,\eta)L_F(T^\alpha-t^\alpha)\alpha^{-1})} \\
\times&\left[\mathcal{C}_1(N,\eta)\mathcal{C}_2(N,q)(\mathcal{N}_{tr})^{-\frac{Nq-2N}{4q}}\exp\left((\mathcal{N}_{tr})^{s-1}b^{-1}T^\alpha\alpha^{-1}\right)\epsilon+(\mathcal{N}_{tr})^{-\eta_0}\,\mathcal{M}\right].
\end{aligned} \tag{38}
$$

In view of the Sobolev embedding $\mathbb{X}^\eta(\mathcal{D})\hookrightarrow L^{\frac{2N}{N-4\eta}}(\mathcal{D})$, we have that

$$
\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{L^{\frac{2N}{N-4\eta}}(\mathcal{D})} \le \mathcal{C}_4(N,\eta)\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{\mathbb{X}^\eta(\mathcal{D})}. \tag{39}
$$

Also

$$
\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{\mathbb{X}^\eta(\mathcal{D})} \le \underbrace{\left\|\widetilde{U}_\epsilon(\cdot,t)-\mathbb{T}_{\mathcal{N}_{tr}}u(\cdot,t)\right\|_{\mathbb{X}^\eta(\mathcal{D})}}_{\mathcal{S}_1}+\underbrace{\|u(\cdot,t)-\mathbb{T}_{\mathcal{N}_{tr}}u(\cdot,t)\|_{\mathbb{X}^\eta(\mathcal{D})}}_{\mathcal{S}_2}. \tag{40}
$$

Now, the first term $\mathcal{S}_1$ is estimated by

$$
\mathcal{S}_1 \le \left(\sum_{n=1}^{\lambda_n\le\mathcal{N}_{tr}}\lambda_n^{2\eta}\Big(\int_{\mathcal{D}}\left(\widetilde{U}_\epsilon(x,t)-u(x,t)\right)e_n(x)dx\Big)^2\right)^{\frac{1}{2}} \le (\mathcal{N}_{tr})^\eta\left\|\widetilde{U}_\epsilon(\cdot,t)-u(\cdot,t)\right\|_{L^2(\mathcal{D})}. \tag{41}
$$

Note $0 < \eta < \eta_0$ and $\mathcal{S}_2$ can be bounded as

$$\mathcal{S}_2 = \left(\sum_{\lambda_n > \mathcal{N}_{tr}} \lambda_n^{-2a} \lambda_n^{2a} \lambda_n^{2\eta} \left| \langle u(t), e_n \rangle \right|^2 \right)^{\frac{1}{2}} \leq \left(\mathcal{N}_{tr}\right)^{-a} \|u\|_{L^\infty(0,T;\mathbb{X}^{a+\eta}(\mathcal{D}))}. \tag{42}$$

Combining (40) to (42), we conclude that

$$\begin{aligned}
&\left\|\widetilde{U}_\epsilon(\cdot,t) - u(\cdot,t)\right\|_{L^{\frac{2N}{N-4\eta}}(\mathcal{D})} \\
&\leq \mathcal{C}_4(N,\eta) \left\|\widetilde{U}_\epsilon(\cdot,t) - u(\cdot,t)\right\|_{\mathbb{X}^\eta(\mathcal{D})} \leq \mathcal{C}_4(N,\eta)\left(\mathcal{N}_{tr}\right)^{-a} \|u\|_{L^\infty(0,T;\mathbb{X}^{a+\eta}(\mathcal{D}))} \\
&\quad + \mathcal{C}_4(N,\eta)\, e^{-\frac{t^\alpha}{\alpha}(\mathcal{N}_{tr})^{s-1} b^{-1}}\, e^{(1+b\mathcal{N}_{tr})^{-1}(\mathcal{C}_3(N,\eta) L_F (T^\alpha - t^\alpha)\alpha^{-1})} \\
&\quad \times \left[\mathcal{C}_1(N,\eta)\mathcal{C}_2(N,q)(\mathcal{N}_{tr})^{\eta - \frac{Nq-2N}{4q}} \exp\left((\mathcal{N}_{tr})^{s-1} b^{-1} T^\alpha \alpha^{-1}\right)\epsilon + (\mathcal{N}_{tr})^{\eta-\eta_0} \mathcal{M} \right].
\end{aligned} \tag{43}$$

□

5. Simulation

In this section, we present some examples illustrating the theoretical part that we gave above. We recall the problem with some selected input data as follows

$$\frac{{}^{\mathscr{C}}\partial^\alpha}{\partial t^\alpha} w(x,t) - \Delta w(x,t) - \frac{{}^{\mathscr{C}}\partial^\alpha}{\partial t^\alpha} \Delta w(x,t) = F(x,t,w(x,t)), \quad (x,t) \in (0,\pi) \times (0,1), \tag{44}$$

with the following Dirichlet boundary condition

$$w(0,t) = w(\pi,t) = 0, \quad t \in (0,1), \tag{45}$$

adding the final value condition

$$w(x,1) = g(x), \quad x \in (0,\pi), \tag{46}$$

where we choose the constants $T = 1$, $s = 1$, $b = 1$ and $\alpha \in (0,1)$. Next, we set up some tools to support the calculation as follows.

Part 1. We approximate an integral by the composite Simpson rule as follows.

$$\int_{a_i}^{a_{i+1}} \mathcal{G}(s)\,\mathrm{d}s \approx \frac{\Delta\tau}{3} \sum_{m=1}^{M/2} \left[\mathcal{G}(s_{2m-2}) + 4\mathcal{G}(s_{2m-1}) + \mathcal{G}(s_{2m}) \right], \tag{47}$$

where $s_m = a_i + m\Delta\tau$ for $m = 1, \ldots, M-1, M$ and $\Delta\tau = \dfrac{a_{i+1} - a_i}{M}$.

Part 2. A uniform grid of mesh points (x_m, t_k) is employed to discretize the space and time intervals

$$x_m = \frac{(m-1)\pi}{M}, \; m = 1,2,\ldots,M+1, \; t_k = \frac{k-1}{N}, \; k = 1,2,\ldots,N+1. \tag{48}$$

The solution $u(x,t)$ can be presented in matrix form as follows

$$\begin{bmatrix}
\mathcal{U}_{1,1} & \mathcal{U}_{1,2} & \cdots & \mathcal{U}_{1,N+1} \\
\mathcal{U}_{2,1} & \mathcal{U}_{2,2} & \cdots & \mathcal{U}_{2,N+1} \\
\vdots & \vdots & \vdots & \vdots \\
\mathcal{U}_{M+1,1} & \mathcal{U}_{M+1,2} & \cdots & \mathcal{U}_{M+1,N+1}
\end{bmatrix}_{(M+1)\times(N+1)},$$

$\{t,\varepsilon\}$	$\alpha=0.1$		$\alpha=0.5$		$\alpha=0.9$	
	$RE^{\varepsilon}_{\alpha}(t)$	$PE^{\varepsilon}_{\alpha}(t)$	$RE^{\varepsilon}_{\alpha}(t)$	$PE^{\varepsilon}_{\alpha}(t)$	$RE^{\varepsilon}_{\alpha}(t)$	$PE^{\varepsilon}_{\alpha}(t)$
$t=0.2,\varepsilon=1\times10^{-1}$	36.37	1.44 %	76.35	2.6 %	18.5	0.79 %
$t=0.2,\varepsilon=3\times10^{-2}$	25.07	0.99 %	52.92	1.81 %	12.75	0.55 %
$t=0.2,\varepsilon=5\times10^{-3}$	12.97	0.51 %	27.54	0.94 %	6.59	0.28 %

TABLE 1. The table error for $\alpha\in\{0.1,0.5,0.9\}$, $\varepsilon\in\{1\times10^{-1},3\times10^{-2},5\times10^{-3}\}$ and $t=0.2$.

where the elements $\mathcal{U}_{m,n}$ are given by

$$\mathcal{U}_{m,n} := u(x_m,t_k) = \sqrt{\frac{2}{\pi}}\sum_{n=1}^{\mathcal{N}} U_n(t_k)\sin(nx_m)$$
$$= \sqrt{\frac{2}{\pi}}\begin{bmatrix} U_1(t_k) & U_2(t_k) & \cdots & U_{\mathcal{N}}(t_k)\end{bmatrix}\times\begin{bmatrix}\sin(x_m)\\ \sin(2x_m)\\ \vdots \\ \sin(\mathcal{N}x_m)\end{bmatrix}, \tag{49}$$

where $\mathcal{N}$ is a parameter truncation,

$$U_n(t_k) = \exp\left(\lambda_n^s(1+b\lambda_n)^{-1}(1-t_k{}^{\alpha})\alpha^{-1}\right)\Big(\int_0^{\pi} g(s)e_n(s)ds\Big)$$
$$-(1+b\lambda_n)^{-1}\left(\int_{t_k}^{1}\mu^{\alpha-1}\exp\left(\lambda_n^s(1+b\lambda_n)^{-1}(\mu^{\alpha}-t_k{}^{\alpha})\alpha^{-1}\right)\left(\int_0^{\pi}F(s,\mu,w(s,\mu))e_n(s)ds\right)d\mu\right). \tag{50}$$

Part 3. The relative error estimation $RE^{\varepsilon}_{\alpha}$ and percent error estimation are $PE^{\varepsilon}_{\alpha}$

$$RE^{\epsilon}_{\alpha}(t) = \sum_{i=1}^{M+1}\Big|u_{\alpha}(x_i,t)-u^{\epsilon}_{\alpha}(x_i,t)\Big|,\ PE^{\epsilon}_{\alpha}(t) = RE^{\alpha'}_{\alpha}(t)\Big/\sum_{i=1}^{M+1}\Big|u_{\alpha}(x_i,t)\Big|\times 100. \tag{51}$$

In this example, we choose the input data

$$\begin{cases} F(x,t,w) = w^2-\left[16(t^{2\alpha}+\alpha)+34\alpha t^{\alpha}+(t^{4\alpha}+2\alpha t^{2\alpha}+\alpha^2)\right]\sin(4x), & (x,t)\in(0,\pi)\times(0,1),\\ g(x) = (1+\alpha)\sin(4x), & x\in(0,\pi). \end{cases} \tag{52}$$

Let $\mathcal{N}=10, M=N=100$, and we have the following result

Tables 1,2, 3 and Figure 1, we see that the numerical result is showing the convergent estimate of the relative error estimation RE^{ϵ}_{α} and percent error estimation PE^{ϵ}_{α} between the solutions. In this example, we consider three cases of ϵ such as $\epsilon\in\{1\times10^{-1},3\times10^{-2},5\times10^{-3}\}$ at $t\in\{0.2,0.4,0.6\}$. In Figure 1, we also show the 3D-graph of the solutions u on the domain $(x,t)\in[0,\pi]\times[0,1]$.

$\{t,\varepsilon\}$	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
	$RE_\alpha^\varepsilon(t)$	$PE_\alpha^\varepsilon(t)$	$RE_\alpha^\varepsilon(t)$	$PE_\alpha^\varepsilon(t)$	$RE_\alpha^\varepsilon(t)$	$PE_\alpha^\varepsilon(t)$
$t = 0.4, \varepsilon = 1\times 10^{-1}$	44.13	1.5 %	19.6	0.72 %	8.25	0.32 %
$t = 0.4, \varepsilon = 3\times 10^{-2}$	30.64	1.04 %	13.59	0.50 %	5.77	0.22 %
$t = 0.4, \varepsilon = 5\times 10^{-3}$	15.97	0.54 %	7.07	0.26 %	3.02	0.12 %

TABLE 2. The table error for $\alpha \in \{0.1, 0.5, 0.9\}$, $\varepsilon \in \{1\times 10^{-1}, 3\times 10^{-2}, 5\times 10^{-3}\}$ and $t = 0.4$.

$\{t,\varepsilon\}$	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
	$RE_\alpha^\epsilon(t)$	$PE_\alpha^\epsilon(t)$	$RE_\alpha^\epsilon(t)$	$PE_\alpha^\epsilon(t)$	$RE_\alpha^\epsilon(t)$	$PE_\alpha^\epsilon(t)$
$t = 0.6, \varepsilon = 1\times 10^{-1}$	9.65	0.35 %	2.1	0.08 %	2.39	0.09 %
$t = 0.6, \varepsilon = 3\times 10^{-2}$	7.01	0.26 %	1.15	0.04 %	4.92	0.18 %
$t = 0.6, \varepsilon = 5\times 10^{-3}$	3.81	0.14 %	0.44	0.02 %	2.39	0.09 %

TABLE 3. The table error for $\alpha \in \{0.1, 0.5, 0.9\}$, $\varepsilon \in \{1\times 10^{-1}, 3\times 10^{-2}, 5\times 10^{-3}\}$ and $t = 0.6$.

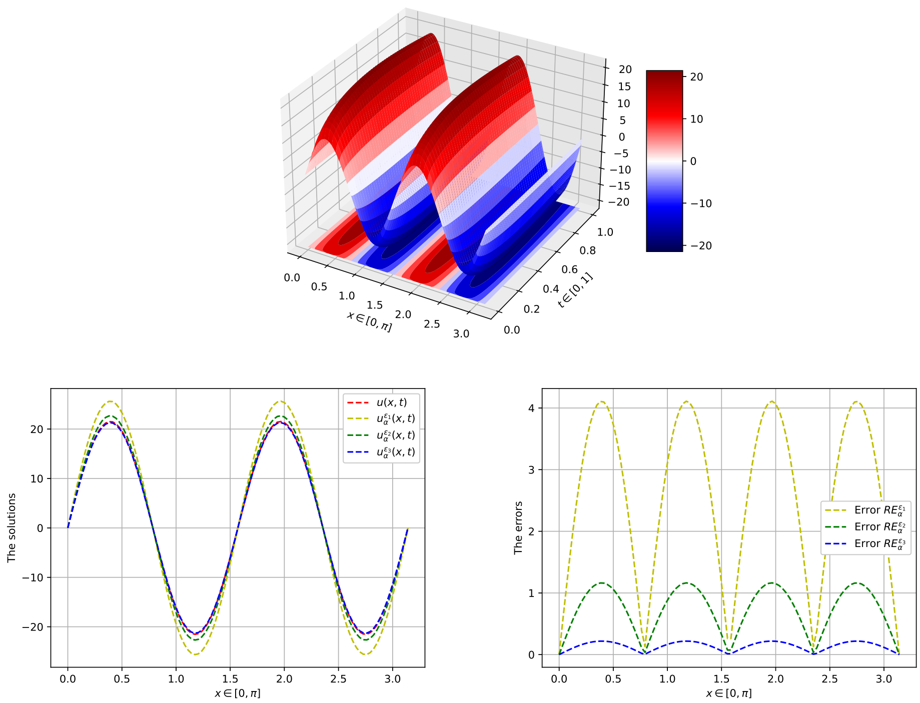

FIGURE 1. An example figure

6. Conclusion

In this work, by applying the Fourier series truncation method, we introduced the regularized solution, and presented the error between the regularized solution and the exact solution derived in the space $L^q(\mathcal{D})$. We illustrated this with appropriate examples.

7. Declarations

Competing Interests

The author(s) declare that they have no competing interests

Ethical Approval

Not applicable

Availability Data and Materials

Not applicable

Author's Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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