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The continuity of solution set of a multi-point boundary problem with a control system

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Abstract

In this paper, we prove the unbounded continuity of the positive set of the inclusion $x \in A \circ T(\lambda, x)$ and apply it to the problem that finds (λ, u) satisfying

$$\begin{cases} D^2u(t) + q(\lambda,t)f(t,u(t)) = 0,\ t \in (0,1), \\ q(\lambda,t) \in F(\lambda,u(t)) \text{ a.e. } t \in [0,1] \\ u(0) = 0 \text{ (resp., } Du(0) = 0), u(1) = \sum_{j=1}^{m} \gamma_j u(\eta_j). \end{cases} \tag{0.1}$$

Here, D^n is the derivative of n order ($D \equiv D^1$). To obtain results, we use topological degree theories and the monotone lower evaluation for multivalued mapping in the infinite neighborhood of λ.

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1 Introduction

The problem of finding x satisfying $x = \mathcal{T}(\lambda, x)$ in ordered spaces has been studied by mathematicians researchers and has found many interesting results (see [3, 7, 23]). It was generalized to the multivalued form

$$x \in \mathcal{T}(\lambda, x). \tag{1.1}$$

There are numerous good method approaches, one of which is the principal eigenvalue - eigenvector method (see impressive results of J. R. L. Webb and K. Q. Lan in [23], Guy Degla in [3]), monotone minorant method [9, 10], the method of using the definition of topological degree (the fixed point index) for single-valued/multivalued mappings [1, 8, 17, 19–22] and the method of combining two latter methods [10]. The solution set of (1.1) is well-known in two following forms

$$S = \{(\lambda, x) : x \in \mathcal{T}(\lambda, x)\}, \tag{1.2}$$

and

$$S = \{x : \exists \lambda = \lambda(x), x \in \mathcal{T}(\lambda, x)\}. \tag{1.3}$$

In the direction where the solution set of (1.1) is considered in the form (1.2), Krasnoselskii [13] used the monotone minorant method in conjunction with the theory of fixed point index to prove that the set S forms an unbounded continuous branch emanating from zero, i.e, $S \cap \Omega \neq \emptyset$ for all bounded open neighborhood Ω of zero. Further, when $\mathcal{T}(\lambda, x) \equiv \lambda \mathcal{T}(x)$, in the Banach space ordered by a cone he obtained the existence of an interval of λ for the equation $x = \lambda \mathcal{T}(x)$.

In the work [23], Webb and Lan established for the existence of multiple positive solutions of a Hammerstein integral equation of the form

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))ds \equiv Au(t),$$

here, the conditions were determined by the relationship between the value of $\frac{f(t,u)}{u}$ as u tends to 0^+ or ∞ and the principal positive-eigenvalue of the linear operator

$$\mathcal{L}u(t) = \int_0^1 k(t,s)g(s)u(s)ds.$$

The result they got was the existence of positive solutions of a second-order differential equation

$$u''(t) + g(t)f(t,,u(t)) = 0 \ \text{ a.e on } \ [0,1]. \tag{1.4}$$

This direction was later developed by several authors, including [4, 12], there the equation was studied with higher derivative operators

$$u^{(2n)}(.) = h(.)f(.,u(.),u^{(1)}(.)....,u^{(2(n-1))}(.)); \tag{1.5}$$

$$Lu(t) = \lambda q(t)u(t), \text{ here } Lu(t) := u^{(n)} + p_1(t)u^{(n-1)} + ... + p_n(t)u. \tag{1.6}$$

Using the eigenvalue criteria, Jiang [12] studied the existence of multiple positive solutions for the integral equation $u(t) = \int_0^1 k(t,s)h(s)f[s, \mathcal{A}_{n-1}u(s), \mathcal{A}_{n-2}u(s), ..., u(s)]$ and applied to the equation (1.5). In [4], by the maximum principle in [2] and a result on the principal eigenvalue of multi-point Boundary Value Problems in [3], Degla proved that the conjugate multi-point boundary value problem (1.6) has non-trivial solutions.

Huy [9] used the topological degree for a completely continuous single-valued operator combined with a monotone minorant to study the unbounded continuity of the solution set of the form (1.3) of equations $x = f(\lambda, x)$, $x = A \circ f(\lambda, x)$.

Recently, the authors of [4, 12, 18, 20] investigated the inclusion (1.1) and proved the unbounded continuity of the solution set (1.3), and they applied it to the multi-point boundary problem (0.1) under conditions $0 < \eta_j < 1$, $\gamma_j \geq 0$, $\sum_{i=j}^m \gamma_j \eta_j < 1$ (resp., $\sum_{i=j}^m \gamma_j < 1$) . The authors of [18] used the fixed point index of multivalued mappings in conjunction with a monotone minorant and continuous embedding from C^1 into $\mathcal{C}$ to proved that the solution set of (1.1) is unbounded continuous branch emanating from 0. In [20], when the authors considered the limits $\overline{\lim}_{\|x\|\to 0}\lambda(x)$ and $\underline{\lim}_{\|x\|\to\infty}\lambda(x)$ (resp., $\overline{\lim}_{\|x\|\to\infty}\lambda(x)$ and $\underline{\lim}_{\|x\|\to 0}\lambda(x)$), they got an eigenvalue interval of λ such that the set of positive solutions of the inclusion $x \in \lambda T(x)$ is nonempty.

Some feedback control systems are described by the mathematical model (0.1), where f is the system's dynamic feature, $u(t)$ is the system's state at the moment t, and $q(t)$ is the control parameter that can be selected from the control set $F(\lambda, u(t))$ depending on the system's state and the parameter λ. Many studies with interesting results have been conducted in this model when $q = F$ is a single-valued function independent of the parameter, e.g, [11, 14, 16, 18]. These authors considered the existence of a solution set of the problem in the form (1.1).

Our current article is mainly inspired by the works [4, 9, 12, 18, 20]. We study the unbounded continuity of the solution set of the form (1.3) of the inclusion $x \in A \circ \Phi(\lambda, x)$ by using the topological order combined with a monotonic lower bound in the infinite neighborhood of λ.

By the abstract result obtained, we apply for the control problem with second-order derivative and multi-point boundary conditions (1.1). Several researchers are becoming more interested in such issues. We will solve this problem to illustrate the method. We also get the same result in [18, 20] but with weaker conditions.

The paper is organized as follows. In the next section, we recall some notations and useful lemmas for representing our results which are stated in sections 3 and 4.

2 Preliminaries

Let $(E, \|.\|)$ be a real Banach space and $\emptyset \neq K \subset E$. K is said to be a cone if $\lambda K \subset K$ for $\lambda \geq 0$, $K \cap (-K) = \{0\}$. An ordering in E by $x \leq y$ iff $y - x \in K$ for $x, y \in X$.

Definition 2.1. For nonempty subsets A, B of E we define

- $A \geq_2 B$ (or, $B \leq_2 A$) iff for every $x \in A$, there exists $y \in B$ satisfying $x \geq y$ (or, $y \leq x$).
- $A \leq_1 B$ iff for every $x \in A$, there exists $y \in B$ such that $x \leq y$.

The cone K is called normal if there exists a constant $N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq N\|y\|$. If $(E, \|.\|)$ is a Banach space with the ordering by the cone K, then there exist a norm $\|.\|_*$ in E such that $\|x\|_* \leq \|y\|_*$ for all $x, y \in K$ with $0 \leq x \leq y$. Further, $\|.\| \sim \|.\|_*$ if K is normal [1]. Throughout this article we always assume that K is normal cone with $N = 1$.

For $W \subset E$, the all nonempty closed (resp., convex closed, compact and convex) subsets of W is denoted by $c(W)$ (resp., $cc(W)$, $kc(W)$). Denote $\mathbb{R}_+ = [0, \infty)$, $R_{++} = \mathbb{R}_+ \backslash \{0\}$. Let Ω be an open subset of E, $\Omega_K = K \cap \Omega$, $\partial_K\Omega = K \cap \partial\Omega$ and $\dot{K} = K \backslash \{0\}$, where $\partial\Omega$ is the boundary of Ω in E.

Definition 2.2. Let $T : K \cap \overline{\Omega} \to cc(K)$ be a multi-valued mapping. T is called

- compact iff $T(W)$ is relatively compact for any bounded subset W of $K \cap \overline{\Omega}$, where $T(W) = \cup_{x \in W} T(x)$;
- upper semicontinuous (in short, u.s.c.) if $\{x \in K \cap \overline{\Omega} : T(x) \subset W\}$ is open in $K \cap \overline{\Omega}$ for every open subset W of K.

As well known (in [8]), if $x \notin T(x)$ for all $x \in \partial_K\Omega$, the fixed point index of T in Ω with respect to K is defined which is an integer denoted by $i_K(T, \Omega)$.

We review the results using to represent our main results.

1 M. Krein, Proprietes fondamentable des ensembles coniques normaux dans l'espace de Banach, Dokl. Acad. Sci. URSS, 28(1940), 13-17.

Lemma 2.3. ([6, Chapter 4, Proposition 20]) *Let E be a Banach space and $F : W \subset E \to c(X)$.*

- *Assume that F is an u.s.c. multivalued mapping in W and any a net $x_\alpha \to x$, $y_\alpha \in F(x_\alpha)$, and $y_\alpha \to y$. Then $y \in F(x)$.*
- *If F is compact and its graph is a closed set in $W \times E$, then F is u.s.c. in W.*

Denote by $\mathcal{C}(J)$ (resp., $\mathcal{L}^1(J)$) the real Banach spaces of all continuous (resp., integrable Lebesgue) functions from J to $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)| : t \in J\}$ (resp., $\int_J |x(t)|dt$).

We also need the following lemma from [15]. This result has been presented in [5, Lemma 4.4, 4.5].

Lemma 2.4. *Let $J = [0,1]$, $L : \mathcal{L}^1(J) \to \mathcal{C}(J)$ be a continuous linear mapping, and $F : J \times \mathbb{R} \to kc(\mathbb{R})$ be an upper-Caratheodory. Then*

$$\mathcal{F}_u(\lambda) := \left\{ x \in \mathcal{L}^1(I) : x(t) \in F(\lambda, u(t)) \ a.e \ t \in J \right\} \neq \emptyset$$

for $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{C}(I)$ and the graph of the multivalued mapping $u \mapsto L(\mathcal{F}_u(\lambda))$ is a closed subset in $\mathcal{C}(I) \times \mathcal{C}(J)$ for all $\lambda \in \mathbb{R}_+$.

Definition 2.5. We say that the set S is an unbounded continuous branch emanating from 0, if $S \cap \partial G \neq \emptyset$ for every bounded open subset $G \ni 0$.

Lemma 2.6. [8, Theorem 2.1] *Assume that multivalued mapping $H : [0,1] \times K \cap \overline{\Omega} \to cc(K)$ is u.s.c. compact satisfying $x \notin H(t,x)$ for all $(t,x) \in [0,1] \times \partial_K\Omega$. Then, $i_K(H(0,.),\Omega) = i_K(H(1,.),\Omega)$.*

The following lemma on the computation of the index were taken in [8, proof of Theorem 3.2].

Lemma 2.7. [8, 10, proof of Theorem 3.2] *Let $T : K \cap \overline{\Omega} \to cc(K)$ be an u.s.c. compact multivalued mapping and $x \notin T(x)$ for all $x \in K \cap \partial\Omega$. Then*

- *$i_K(T,\Omega) = 0$ if there is $u \in \dot{K}$ satisfying $x \notin T(x) + ku$ for all $x \in \partial_K\Omega$ and $k > 0$.*
- *$i_K(T,\Omega) = 1$ if $0 \in \Omega$ and $kx \notin T(x)$ for all $k > 1$.*

We recalled the under corollary established in [21] (by use the above lemmas (Lemma 2.6, 2.7 and Part 1 of Lemma 2.3)) to aid in the presentation of the main result in the next section.

Lemma 2.8. [21, Lemma 4.1] *Let $T : [0,\infty) \times K \to cc(K)$ be an u.s.c compact operator and $\Omega \ni 0$ be an open bounded subset of E. Assume that the following conditions are satisfied*

- *$tx \in T(0,x)$ for some $x \in \dot{K}$ implies $t < 1$,*
- *there exists $\lambda_0 > 0$ such that $i_k(T(\lambda,.),\Omega) = 0$ for all $\lambda \geq \lambda_0$.*

Then the set $\{x \in \partial_K\Omega : \exists\lambda > 0, x \in T(\lambda,x)\}$ is nonempty.

3 Abstract results

Let $T : \mathbb{R}_+ \times K \to cc(K)$. The opperator $\phi : K \to K$ is said to be a monotone ∞-minorant of the operator T if ϕ is continuous and there exists $\lambda_* \geq 0$ such that $T(\lambda,x) \geq_2 \phi(\lambda x)$ for all $(\lambda,x) \in [\lambda_*,\infty) \times K$.

Theorem 3.1. *Assume that $T : (\mathbb{R}_+, K) \to cc(K)$ be an u.s.c compact multi-valed mapping and the following conditions are satisfied*

- *$tx \in T(0,x)$ for some $x \in \dot{K}$;*
- *the cone K is normal, the operator T has a monotone minorant ϕ; there are positive numbers $a > 0$, $b > 0$ and $u \in \dot{K}$ such that*
  - *$\phi(\tau u) \geq a\tau u$ for all $\tau \in [0,b]$;*
  - *$\lim_{\tau\to\infty} \|\phi(\tau u)\| = \infty$.*

Then, $S = \{x \in \dot{K} : \exists\lambda > 0, x \in T(\lambda, x)\}$ is unbounded continuous branch emanating from 0.

Proof. Assume that $\Omega \ni 0$ is an open bounded subset of E. Clearly, the first condition of Lemma 2.8 holds. To shows condition 2 of the lemma we will check the following statement:

$$\exists\lambda_0, \forall\lambda \geq \lambda_0, k > 0, x \in \partial_K\Omega, x \notin T(\lambda, x) + ku. \tag{3.1}$$

Suppose that the assertion is false, we can find the sequences $\{\lambda_n\} \subset \mathbb{R}_{++}$, $\{x_n\} \subset \partial_K\Omega$, $\{k_n\} \subset \mathbb{R}_+$ such that

$$\lambda_n \to \infty \text{ and } x_n \in T(\lambda_n, x_n) + k_n u. \tag{3.2}$$

Let $\mu_n = \sup\{\mu > 0 : x_n \geq \mu u\}$. It folows that $\mu_n > 0$ from (3.2) (since $\mu_n \geq k_n > 0$). Combining (3.2), the condition (a), and the monotone minorant of T we have

$$x_n \geq \phi(\lambda_n x_n) \geq \phi(\lambda_n \mu_n u). \tag{3.3}$$

We denote

$$\mathscr{A} = \{n \in \mathbb{N} : \lambda_m\mu_n \leq b\} \text{ and } \mathscr{B} = \{n \in \mathbb{N} : \lambda_n\mu_n > b\}.$$

If $\mathscr{A}$ (resp, $\mathscr{B}$) has infinal elements, By choosing subsequences, we get from condition 2(b) (resp., 2(a), implies $x_n \geq abu$), (3.3) and the normality property of the cone K that $\{\|x_n\|\}$ is unbounded. This contradicts $x_n \in \partial_K\Omega$. Therefore, (3.1) holds, hence, we obtain $i_K(T(\lambda,.),\Omega) = 0$ for all $\lambda \geq \lambda_0$ by applying Lemma 2.8. The theorem is proved. □

4 Applications

Let $F : [0,\infty) \times \mathbb{R}_+ \to cc(\mathbb{R}_+)$ be a compact multivalued mapping, $J = \{\gamma \in \mathbb{R} : 0 \leq \gamma \leq 1\}$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. We consider control problem which contains a parameter of the form

$$\begin{cases} D^2u(t) + q(\lambda,t)f(t,u(t)) = 0, \ t \in (0,1), \\ q(\lambda,t) \in F(\lambda, u(t)) \text{ a.e. } t \in [0,1] \\ u(0) = 0 \text{ (resp., } D(0) = 0), u(1) = \sum_{j=1}^m \gamma_j u(\eta_j) \end{cases} \tag{4.1}$$

where, $0 < \eta_j < 1$, $\gamma_j \geq 0$, and $\sum_{j=1}^m \gamma_j\eta_j < 1$ (resp., $\sum_{j=1}^m \gamma_j < 1$). The problem (4.1) is considered under the following assumptions

(C1): $f : J \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous,

(C2): $F : \mathbb{R}_+ \times \mathbb{R}_+ \to kc(\mathbb{R}_+)$ for every $(x,y) \in \mathbb{R}_+ \times \mathbb{R}$, $t \mapsto \sup_{a\in F(t,x)}|y-a|$ is a measurable function,

(C3): for a.e t, the function $x \mapsto F(t,x)$ is u.s.c.,

(C4): for each $r > 0$ there exists $\varphi_r \in \mathcal{L}^1(J,\mathbb{R})$ satisfying $\sup_{x\in[0,r]} |||F(t,x)||| \leq \varphi_r(t)$ a.e on J, where $\mathcal{L}^1(J,\mathbb{R})$ is the space of all measurable real-valued functions whose absolute value is Lebesgue integrable on J, and $|||F(t,x)||| = \sup_{u\in F(t,x)} |u|$.

Denote $\Lambda = \sum_{j=1}^{m} \gamma_j \eta_j$ (resp., $\Lambda = \sum_{j=1}^{m} \gamma_j$). for every $(t,s) \in [0,1] \times [0,1]$, we define

$$h(t,s) = \begin{cases} s(1-t), s \leq t, \\ t(1-s),\ s > t \end{cases}$$

$$(\text{resp., } h(t,s) = \begin{cases} 1-t, \quad s \leq t, \\ 1-s, s > t \ ). \end{cases}$$

The Green function of the problem (4.1) is defined by (see, [16])

$$G(t,s) = \frac{t}{1-\Lambda} \sum_{i=1}^{m} \gamma_i h(\eta_i, s) + h(t,s);$$

The real Banach spaces $\mathcal{C}(J)$ endowed the order induced by the cone K of nonnegative functions. For $(\lambda, u) \in \mathbb{R}_+ \times K$, we define

$$\mathcal{F}_u(\lambda) = \{x \in \mathcal{L}^1(J) : x(t) \in F(\lambda, u(t)) \ \text{ a.e } t \in J\}.$$

Lemma 4.1. [20, Lemma 4.3] *Assume the conditions* (C1)-(C4) *and* $(\lambda, u) \in \mathbb{R}_+ \times K$. *We have the following assertions*

- $\mathcal{F}_u(\lambda)$ *is a nonempty closed convex subset in* $L^1[0,1]$,
- $\int_0^1 G(.,s)\, x(s)\, f(s, u(s))\, ds \in K$ *for* $x \in \mathcal{F}_u(\lambda)$.

Let $L : \mathcal{L}^1(J) \to \mathcal{C}(J)$ be a linear operator defined by

$$L(u)(t) = \int_0^1 G(t,s) u(s) ds \text{ for all } t \in J. \tag{4.2}$$

Clearly, L is well defined by Lemma 4.1. We obtain that L is compact by using [20, Lemma 4.2]. Let Φ be a multivalued operator defined by

$$\begin{aligned} \Phi(\lambda, x(t)) &= F[\lambda, x(t)] f[t, x(t)], \\ &= \{q(t).f(t,x(t)) : q \in \mathcal{F}_x(\lambda)\}, \text{ a.e } t \in J, \text{ for all } (\lambda, x) \in \mathbb{R}_+ \times K. \end{aligned}$$

We also see from the conditions (c1)-(c2) that the function $s \mapsto G(t,s)q(s)f(s,x(s))$ is integrable Lebesgue on J for all $x \in K$, $t \in J$, $\lambda \in \mathbb{R}_+$ with $q \in \mathcal{F}_x(\lambda)$. Hence, this leads to $\Phi(\lambda, x) \subset \mathcal{L}^1(J)$ for all $(\lambda, x) \in \mathbb{R}_+ \times K$.

We define $A = L|_K$, i.e, $A(u) = L(u)$ for all $u \in K$, and we have $A(K) \subset K$.

It is clear that (4.1) has a positive solution iff there exist $x \in \dot{K}$, $\lambda \in \mathbb{R}_+$ and a function $q(\lambda, .)$ satisfying

$$q(t) \in F(\lambda, x(t)) \text{ and } x(t) = \int_0^1 G(t,s) q(s) f(s, x(s)) ds \ \text{ for a.e } t \in J.$$

Therefore, instead of solving problem (4.1) we shall consider its equivalent form

$$x \in A \circ \Phi(\lambda, x), \tag{4.3}$$

From the well-known results in [23], the compact linear operator A has an eigen-value $\mu_0 > 0$ with respect to a positive eigen-function $u_0 \in \text{int}K$. Assume conditions (c1)-(c4). Using the lemmas 2.3 and 2.4 and the argument similar to the proof of Proposition 4.4 in [20], we deduce that $T(\lambda, .)$, defined by $T(\lambda, x) := A \circ \Phi(\lambda, x)$, is u.s.c compact with closed convex values for all $\lambda \in \mathbb{R}_+$.

Theorem 4.2. *Assume conditions (C1)-(C4) and there exist $\lambda_* > 0$ and the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions*

- $F(0,s) = \{0\}$;
- $F(\lambda,s)f(s) \geq_2 \{g(\lambda s)\}$ *for all* $(\lambda,s) \in [\lambda_*,\infty) \times \mathbb{R}_+$;

Then, the set $\{x \in K : \exists\lambda > 0, x \in A \circ \Phi(\lambda,x)\}$ forms an unbounded continuous branch in $C(J)$, emanating from 0.

Proof. By We shall apply Theorem 3.1 with the multi-valued mapping T. Condition 1 of the theorem is self-evident. Condition 2 of the theorem will be examined. Consider the linear operator $\phi : K \to K$ difined by $\phi(u)(t) = \int_0^1 G(t,s)g(u(s))ds$. Denote $\alpha = \sup\{\mu > 0 : g(u) \geq \mu u_0\}$. Since $u_0 \in \text{int}K$, it implies $\alpha > 0$. From the asumption we have

$$T(\lambda, x(t)) \geq_2 \int_0^1 G(t,s)g(\lambda x(s))ds \ \ \forall(\lambda,t) \in [\lambda_*,\infty) \times J. \tag{4.4}$$

Hence, T has the monotone ∞-mimorant ϕ difined by $\phi(\lambda u)(t) := \int_0^1 G(t,s)g(\lambda u(s))ds$.

$$\phi(\tau u_0)(t) \geq \int_0^1 G(t,s)\alpha\tau u_0(s)ds = \alpha\mu_0\tau u_0(t) \ \forall\tau \in [0,b], t \in [0,1]. \tag{4.5}$$

Since K in the normal cone, it folows that $\lim_{\tau\to\infty} \|\phi(\tau u_0)\| = \infty$ from the condition 2. All assumptions of the Theorem 3.1 are satisfied. The theorem is proved. □

Finally, the following example shows that the assumptions of Theorem 4.2 are fulfilled.

Example 1. Let $F : \mathbb{R}_+ \times \mathbb{R}_+ \to cc(\mathbb{R}_+)$, $f : J \times \mathbb{R}_+ \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $F(\lambda,s) = [\frac{s+1}{s+2}\lambda, \lambda]$, $f(t,s) = \frac{1}{s+1}$ and $g(s) = \frac{s}{2\,(s+1)}$. Then, $T(\lambda,s) \geq_2 g(\lambda s)$ for all $(\lambda,s) \in \left[\frac{1}{2},\infty\right) \times \mathbb{R}_+$. Hence, all the conditions of Theorem 4.2 are satisfied.

5 Conclusion

The unbounded continuity of the positive solution set for a multivalued inclusion containing a parameter has been proved and the result is applied to the feedback control systems. The technique of combining the monotone minorant with the topological degree property is applied in the results.

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