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RESEARCH ARTICLE

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Reconstructing the right-hand side of a Poisson equation with random noise

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Abstract. An inverse source problem for the Poisson equation is looked at in this article. This is a problem that is poorly posed because even minor changes in the data can result in arbitrarily large changes in the results. We first demonstrate some useful lemmas about our proposed problem before presenting the main results. Then, at that point, we propose a regularization strategy to manage the reverse source issue and get a union rate with random noise.

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1. Introduction

Applications in science and engineering rely heavily on inverse source problems (see [1] and [2]). Finding the hidden source's location, size, and shape from the boundary's measured data is the inverse source problem of the Poisson equation's source term determination problem. The nonlinearity and illusory nature of this inverse source problem can be seen in the fact that, even if a solution is found, it does not always rely on the measured data. The given data could have small errors that result in large errors in the solution. As a result, our proposed algorithms should make use of regularization methods. Numerous studies have been conducted on the Poisson equation's inverse source problems ([3], [4], [5], [9], [10], [11]. Bubnov, Erokhin and Isakov [5], [12] introduced a few hypothetical outcomes to recreate the obscure source or impediments from over determined limit estimations of arrangements of the Poisson condition. From over determined boundary measurements of Laplace equation solutions, the inverse problem of locating pointwise or small size conductivity defaults in a plane domain was solved by [4]. Hon and co. Several efficient numerical algorithms for resolving Poisson equation inverse source problems were proposed [11], [15]. In order to identify hidden obstacles, Hanke and Rundell's used the rational approximation method, see in [8]. Iterative algorithms exist to obtain source parameters from boundary measurement data [9], [10], [13], [17], [18]. An inverse potential problem for reconstructing an obstacle's shape was solved using iterative algorithms in Hettlich and Rundell's [10]. In this paper, assume that Ω is a simply connected bounded domain of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, we consider to find a pair of functions $(u(x,y), \mathcal{F}(x))$ satisfying

$$-u_{xx} - u_{yy} = \mathcal{F}(x), 0 < x < \pi, \quad 0 < y < +\infty, \tag{1}$$

with boundary condition

$$u(0,y) = u(\pi,y) = 0, 0 \leqslant y < +\infty, \tag{2}$$

and final condition

$$u(x,1) = h(x), 0 \leqslant x \leqslant \pi. \tag{3}$$

and

$$u(x,0) = 0, \, u(x,y)|_{y\to\infty} \, \text{bounded}, 0 \leqslant x \leqslant \pi, \tag{4}$$

where $u(x,1) = h(x)$ is the supplementary condition and $\mathcal{F}(x)$ is the unknown source that is only dependent on one spatial variable. The inverse problem of identifying an unknown source is the name given to this issue. Numerous studies have been conducted on the various heat source types cited in the heat equation [23], [24], [25], [26], [27], [28]. We are aware of only a few papers that attempted to identify the random noise-based unknown source in the Poisson equation. Since we cannot anticipate that the measured data function $h_\epsilon(x)$ will decay at the same rate in $L^2(0,\pi)$, the problem (1) is ill-posed. The ill-posed problem will be addressed by means of the truncation regularization approach in the following section. The truncation regularization strategies have been read up for reverse issues, with the end goal that : Eldén can be found in [30], Xiong in [31], Fu in [32], and Qian in [33].

The model is random if the errors are brought on by unpredictability, such as wind, rain, humidity, or other factors. The approximate representation of the final data, h, is known as h_ε. The arbitrary model can't be tackled utilizing similar procedures used for the deterministic cases. It is regularly difficult to work out due to the irregular clamor. We have

$$\widetilde{h}(x_k) = h(x_k) + \varepsilon_k, \quad k = 1, \cdots, n.$$

where $\varepsilon_k, k = 1, \cdots, n$ are unknown independent random errors because the function $h(x)$ in practical applications is the result of experimental observations and cannot be viewed without errors. As a matter of fact, these mistakes can emerge out of many sources like the estimating instrument or the climate. From now on, we put $x_k = \pi\frac{2k-1}{2n}$, with $k = 1, \cdots, n$. We have a data set $\mathcal{D} = \left(\widetilde{h}(x_1), \widetilde{h}(x_2), \ldots, \widetilde{h}(x_n)\right)$, which is the measure of $\left(h(x_1), h(x_2), \ldots, h(x_n)\right)$, here $\mathcal{D}$ satisfies

$$\widetilde{h}\big(x_k\big) = h\big(x_k\big) + \sigma_k \varepsilon_k, \tag{5}$$

where, ε_k, $k = 1, \cdots, n$ are unknown independent noises. Therefore, ε_k and σ_k are unknown positive constants that are constrained by the positive constant $\mathcal{V}_m ax$ so that $0 \leq < \sigma_k$ and $\mathcal{V}_{\max}$, respectively. The noises ε_k are independent of one another. There are no published results on the inverse source problem for fractional diffusion with random noise. Our fundamental issue in this paper is finding the source capability $\mathcal{F}$ from the arbitrary information $h(x_k), k = 1, \cdots, n$. For $\sigma > 0$, let $\mathcal{H}^\sigma(\Omega)$ be the arrangement of all capabilities $\mathcal{F} \in L^2(\Omega)$ with

$$\|\mathcal{F}\|^2_{\mathcal{H}^\sigma(\Omega)} = \sum_{p=1}^{\infty} p^{2\sigma} \left|\langle \mathcal{F}, \xi_p \rangle\right|^2 < \infty. \tag{6}$$

The outline of the paper is as follows. In Section 2, we present some preparatory knowledge. The main result is in Section 3.

2. Preminilaries

Lemma 2.1. (See in [7]) If $p \geqslant 1$, we have

$$\left(1 - \exp(-p)\right)^{-1} \leqslant 2. \tag{7}$$

Lemma 2.2. Let $p = 1, \dots, n-1$, and $q = 1, 2, \dots$, with $x_k = \pi \dfrac{2k-1}{2n}$ and $\xi_p(x_k) = \sqrt{\dfrac{2}{\pi}} \sin(px_k)$, then we have

$$\mathcal{S}_{p,q} = \frac{1}{n} \sum_{k=1}^{n} \xi_p(x_k) \xi_q(x_k) = \begin{cases} \dfrac{1}{\pi}, q - p = 2\ell n \text{ or } q + p = 2\ell n (\ell \text{ even }), \\ -\dfrac{1}{\pi}, q - p = 2\ell n \text{ or } q + p = 2\ell n (\ell \text{ odd }), \\ 0, \text{ otherwise.} \end{cases} \tag{8}$$

If $q = 1, 2, \dots, n-1$, then

$$\mathcal{S}_{p,q} = \begin{cases} \dfrac{1}{\pi}, & p = q, \\ 0, & p \neq q, \end{cases} \text{, and } \frac{1}{n} \sum_{k=1}^{n} \xi_p(x_k) = \begin{cases} 0, & p \neq 2\ell n, \\ (-1)^\ell \sqrt{\dfrac{2}{\pi}}, & p = 2\ell n. \end{cases} \tag{9}$$

Lemma 2.3. (See [6]) Let $p, n \in \mathbb{N}$ such that $1 \leq p \leq n-1$, and $h \in C[0, \pi]$. Then we have

$$\langle h, \xi_p \rangle = \frac{\pi}{n} \sum_{k=1}^{n} h(x_k) \xi_p(x_k) - \sum_{\ell=1}^{\infty} (-1)^\ell \Big(\langle h, \xi_{p+2\ell n} \rangle + \langle h, \xi_{-p+2\ell n} \rangle \Big), \quad 1 \leq p \leq n-1, \tag{10}$$

Lemma 2.4. Let $0 < \mathcal{M}_{tr} < n, \mathcal{M}_{tr} \in \mathbb{N}$, assume that h is as in Lemma 2.3, then the source function $\mathcal{F}$ is given by

$$\begin{aligned} \mathcal{F}_{n,\mathcal{M}_{tr}}(x) = & \sum_{p=1}^{\mathcal{M}_{tr}} \frac{p^2}{1 - \exp(-p)} \Bigg(\frac{\pi}{n} \sum_{k=1}^{n} h(x_k) \xi_p(x_k) \\ & - \sum_{\ell=1}^{\infty} (-1)^\ell \Big(\langle h, \xi_{p+2\ell n} \rangle + \langle h, \xi_{-p+2\ell n} \rangle \Big) \Bigg) \xi_p(x) \\ & + \sum_{p=\mathcal{M}_{tr}+1}^{\infty} \frac{p^2}{1 - \exp(-p)} \langle h, \xi_p \rangle \xi_p(x). \end{aligned} \tag{11}$$

Proof The problem (1) has the mild solution:

$$u(x, y) = \sum_{p=1}^{\infty} \frac{1 - \exp(-py)}{p^2} \langle \mathcal{F}, \xi_p \rangle \xi_p, \text{ where } \left\{ \xi_p = \sqrt{\frac{2}{\pi}} \sin px, (p = 1, 2, \dots) \right\}, \tag{12}$$

and $\langle \mathcal{F}, \xi_p \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi \mathcal{F}(x) \sin(px) dx$. We have

$$h(x) = \sum_{p=1}^{\infty} \langle h, \xi_p \rangle \xi_p(x) = \sum_{p=1}^{\infty} \Big(\frac{1-\exp(-p)}{p^2} \Big) \langle \mathcal{F}, \xi_p \rangle \xi_p(x). \tag{13}$$

Therefore, we have

$$\begin{aligned}
\mathcal{F}(x) &= \sum_{p=1}^{\infty} \Big(\frac{p^2}{1-\exp(-p)} \Big) \langle h, \xi_p \rangle \xi_p(x) \\
&= \sum_{p=1}^{\mathcal{M}_{tr}} \frac{p^2}{1-\exp(-p)} \langle h, \xi_p \rangle \xi_p(x) + \sum_{p=\mathcal{M}_{tr}+1}^{\infty} \frac{p^2}{1-e^{-p}} \langle h, \xi_p \rangle \xi_p(x) \\
&= \sum_{p=1}^{\mathcal{M}_{tr}} \frac{p^2}{1-\exp(-p)} \bigg(\frac{\pi}{n} \sum_{k=1}^{n} h(x_k) \xi_p(x_k) - \sum_{\ell=1}^{\infty} (-1)^\ell \Big(\langle h, \xi_{p+2\ell n} \rangle + \langle h, \xi_{-p+2\ell n} \rangle \Big) \bigg) \xi_p(x) \\
&\quad + \sum_{p=\mathcal{M}_{tr}+1}^{\infty} \frac{p^2}{1-\exp(-p)} \langle h, \xi_p \rangle \xi_p(x).
\end{aligned} \tag{14}$$

3. The main results

Theorem 3.1. *Let $\varepsilon > 0$ and $\varepsilon_k \sim N(0,1)$ be independent normal random variables with $k = 1, \cdots, n$ (as mentioned above), then a regularized function $\widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}}$ for $\mathcal{F}$ can be computed as follows*

$$\widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}}(x) = \sum_{p=1}^{\mathcal{M}_{tr}} \Big(\frac{p^2}{1-\exp(-p)} \Big) \frac{\pi}{n} \sum_{k=1}^{n} h(x_k) \xi_p(x_k) \xi_p(x). \tag{15}$$

$\mathcal{M}_{tr}$ is regularization parameters, it gives

$$\mathbb{E} \big\| \widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}} - \mathcal{F} \big\|^2_{L^2(\Omega)} \le 2 \big(\mathcal{M}_{tr} \big)^{-2\sigma} E^2 + 8 \mathcal{M}_{tr}^4 \Big(\frac{\pi^2}{n^2} \mathcal{V}^2_{\max} + \frac{\pi^4}{144} \frac{\|\mathcal{F}\|^2_{L^2(\Omega)}}{n^4} \Big). \tag{16}$$

Let $\mathcal{M}_{tr} = \mathcal{M}_{tr,n}$ such that $0 < \mathcal{M}_{tr} = \mathcal{M}_{tr,n} < n$ and

$$\lim_{n \to +\infty} \frac{\mathcal{M}_{tr}^4}{n} = 0, \tag{17}$$

then

$$\mathbb{E} \big\| \widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}} - \mathcal{F} \big\|^2_{L^2(\Omega)} \text{ is of order } \Big\{ \frac{\mathcal{M}_{tr}^4}{n}, (\mathcal{M}_{tr})^{-2\sigma} \Big\}. \tag{18}$$

Remark 3.1. If we choose $\mathcal{M}_{tr} = n^{\frac{1}{4+2\sigma}}$, with (18), then we have

$$\mathbb{E} \big\| \widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}} - \mathcal{F} \big\|^2_{L^2(\Omega)} \text{ is of order } n^{-\frac{\sigma}{2+\sigma}}. \tag{19}$$

Proof:

Using the inequality $1 - \exp(-p) \le p$, $\forall p > 0$, we receive

$$\big| \langle h, \xi_p \rangle \big| = \Big(\frac{1-\exp(-p)}{p^2} \Big) \big| \langle \mathcal{F}, \xi_p \rangle \big| \le \frac{\|\mathcal{F}\|_{L^2(\Omega)}}{p}. \tag{20}$$

Using (14) and (15), we obtain

$$\widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}}(x) - \mathcal{F}(x)$$

$$= \sum_{p=1}^{\mathcal{M}_{tr}} \left(\frac{p^2}{1-\exp(-p)}\right)\left(\frac{\pi}{n}\sum_{k=1}^{n}\sigma_k\varepsilon_k\xi_p(x_k) + \sum_{\ell=1}^{\infty}(-1)^{\ell}\Big(\langle h,\xi_{p+2\ell n}\rangle + \langle h,\xi_{-p+2\ell n}\rangle\Big)\right)\xi_p(x)$$
$$- \sum_{p=\mathcal{M}_{tr}+1}^{\infty}\left(\frac{p^2}{1-\exp(-p)}\right)\langle h,\xi_p\rangle\,\xi_p(x)\cdot \tag{21}$$

Applying Lemma 2.4, we have

$$\|\widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}} - \mathcal{F}\|^2_{L^2(\Omega)}$$
$$= \sum_{p=1}^{\mathcal{M}_{tr}}\left(\frac{p^2}{1-\exp(-p)}\right)^2\left(\frac{\pi}{n}\sum_{k=1}^{n}\sigma_k\varepsilon_k\xi_p(x_k) + \sum_{\ell=1}^{\infty}(-1)^{\ell}\Big(\langle h,\xi_{p+2\ell n}\rangle + \langle h,\xi_{-p+2\ell n}\rangle\Big)\right)^2$$
$$+ \sum_{p=\mathcal{M}_{tr}+1}^{\infty}\left(\frac{p^2}{1-\exp(-p)}\right)^2|\langle g,\xi_p\rangle|^2. \tag{22}$$

The fact that $\mathbb{E}(\varepsilon_j\varepsilon_l) = 0; (j \neq l)$, and $\mathbb{E}\varepsilon_j = 0; j = 1,2,\dots,n$. One has

$$\mathbb{E}\|\widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}} - \mathcal{F}\|^2_{L^2(\Omega)}$$
$$\leq 2\underbrace{\sum_{p=\mathcal{M}_{tr}+1}^{\infty}\left(\frac{p^2}{1-\exp(-p)}\right)^2|\langle g,\xi_p\rangle|^2}_{\mathcal{J}_1}$$
$$+\underbrace{2\sum_{p=1}^{\mathcal{M}_{tr}}\left(\frac{p^2}{1-\exp(-p)}\right)^2\left(\frac{\pi^2}{n^2}\sum_{k=1}^{n}\sigma_k^2\mathbb{E}\epsilon_k^2 + \underbrace{\left(\sum_{\ell=1}^{\infty}(-1)^{\ell}\Big(\langle h,\xi_{p+2\ell n}\rangle + \langle h,\xi_{-p+2\ell n}\rangle\Big)\right)^2}_{\mathcal{J}_2}\right)}_{\mathcal{J}_3}. \tag{23}$$

We know that $\sum_{\ell=1}^{\infty}\frac{1}{\ell^2} = \frac{\pi^2}{6}$, we obtain

$$\mathcal{J}_2 \leq \sum_{l=1}^{\infty}|\langle h,\xi_{p+2\ell n}\rangle + \langle h,\xi_{-p+2\ell n}\rangle|$$
$$\leq \frac{\|\mathcal{F}\|_{L^2(\Omega)}}{n^2}\left[\sum_{\ell=1}^{\infty}\frac{1}{(p+2\ell n)^2} + \sum_{\ell=1}^{\infty}\frac{1}{(-p+2\ell n)^2}\right] \leq \frac{\pi^2}{12}\frac{\|\mathcal{F}\|_{L^2(\Omega)}}{n^2}. \tag{24}$$

For $p \geq 1$, $1 = p^{2\sigma}p^{-2\sigma}$, we can rewrite $\mathcal{J}_1$ as follows

$$\mathcal{J}_1^2 = \sum_{p=\mathcal{M}_{tr}+1}^{\infty}\left(\frac{p^2}{1-\exp(-p)}\right)^2|\langle h,\xi_p\rangle|^2 = \sum_{p=\mathcal{M}_{tr}+1}^{\infty}p^{2\sigma}p^{-2\sigma}|\langle\mathcal{F},\xi_p\rangle|^2, \tag{25}$$

this leads to

$$\mathcal{J}_1^2 \leq (\mathcal{M}_{tr})^{-2\sigma}E^2. \tag{26}$$

Next, $\mathcal{J}_3$ can be bounded as follows

$$\mathcal{J}_3^2 \leq \sum_{p=1}^{\mathcal{M}_{tr}}\left(\frac{p^2}{1-e^{-p}}\right)^2\left(\frac{\pi^2}{n^2}\sum_{k=1}^{n}\sigma_k^2\mathbb{E}\epsilon_k^2 + \mathcal{J}_2^2\right)^2. \tag{27}$$

Since $\sigma_k \leq \mathcal{V}_{\max}$, using the Lemma 2.1, one has

$$\mathcal{J}_2^2 \leq \sum_{p=1}^{\mathcal{M}_{tr}} 4p^4 \Big(\frac{\pi^2}{n^2} \mathcal{V}_{\max}^2 + \frac{\pi^4}{144} \frac{\|f\|_{L^2(\Omega)}^2}{n^4} \Big) \leq 4\mathcal{M}_{tr}^4 \Big(\frac{\pi^2}{n^2} \mathcal{V}_{\max}^2 + \frac{\pi^4}{144} \frac{\|\mathcal{F}\|_{L^2(\Omega)}^2}{n^4} \Big). \tag{28}$$

Combining (23), (26), and (27), we obtain

$$\mathbb{E}\big\| \widetilde{\mathcal{F}}_{n,\mathcal{M}_{tr}} - \mathcal{F} \big\|_{L^2(\Omega)}^2 \leq 2\big(\mathcal{M}_{tr}\big)^{-2\sigma} E^2 + 8\mathcal{M}_{tr}^4 \Big(\frac{\pi^2}{n^2} \mathcal{V}_{\max}^2 + \frac{\pi^4}{144} \frac{\|\mathcal{F}\|_{L^2(\Omega)}^2}{n^4} \Big). \tag{29}$$

This completes the proof.

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Authors Contributions

All authors read and approved the final version of the manuscript.

Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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