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Time fractional stochastic Navier-Stokes equations driven by fractional Brownian motion

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Abstract

A two-dimensional time fractional stochastic incompressible Navier-Stokes equation driven by fractional Brownian motion is studied with the Hurst parameter $H \in (1/2, 1)$ and time fractional differential operator of order $\alpha \in (0, 1)$ under the Dirichlet boundary condition. Without the requirement of compact parameters, the existence and regularity of the nonlocal stochastic convolution are obtained by combining the estimate on the spectrum of the Stokes operator under a square domain, an upper bound of a class of the generalized Mittag-Leffler functions, and the fractional calculus technique. Moreover, sufficient conditions are provided to ensure the local and global existence and uniqueness of mild solutions, as well as the square integrability and continuity of the solution's paths.

Key words: stochastic Navier-Stokes equation, fractional Brownian motion, fractional differential equations, Mittag-Leffler functions, mild solution

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1 Introduction

The Navier-Stokes equation (NSE) is the objective of much attention in the scientific literature due to its importance in fluid mechanics and turbulence. To be precise, the classical incompressible NSE is given by

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = f, & \text{in } \mathcal{O}, \\ \nabla\cdot \boldsymbol{u} = 0, & \text{in } \mathcal{O}, \\ \boldsymbol{u} = 0, & \text{on } \partial\mathcal{O}, \\ \boldsymbol{u}_0 = \boldsymbol{u}(0,x), & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O}$ is a bounded open domain in $\mathbb{R}^d$ ($d = 2$ or $d = 3$) with a smooth boundary $\partial\mathcal{O}$, $\boldsymbol{u} = (u_1(t,x), u_2(t,x))$ represents the velocity field, $\nu > 0$ is the viscosity coefficient, p denotes the pressure field, $\boldsymbol{u}_0$ is the initial velocity and f is an external force. For instance, Leary [1] proved the existence, uniqueness, and decay rate for the weak solution to NSE (1.1). Motivated by this pioneer work, there is a wide and solid literature to address this issue by many authors, see, among others, Weissler [2], Kato [3], Schonbek [4], Wiegner [5], Borchers and Miyakawa [6], Carpio [7], Bae and Jin [8].

On the other hand, fractional calculus has attracted the attention of many researchers due to the nonlocal character of the fractional differentiation. By introducing the fractional order derivative into NSE (1.1), one can obtain the time fractional Navier-Stokes equation (TFNSE):

$$D_t^\alpha \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = f, \tag{1.2}$$

under the same initial and boundary conditions as in NSE (1.1), where $\alpha \in (0,1)$ is a fixed number representing the order of time fractional differential operator, and D_t^α is the Caputo fractional derivative to be specified later. In particular, Lions [9] posed the question of whether a weak solution of TFNSE (1.2) has fractional derivative with respect to the time. Shinbrot [10] further studied the weak solution of TFNSE (1.2) and conjectured that the weak solution possesses fractional derivatives of order $\alpha \in (0, 1/2]$. Later on, Zhang [11] proved Shinbrot's conjecture by using the classical Hardy-Littlewood maximal theorem. Recently, Carvalho-Neto and Planas [12] proved the existence, uniqueness, decay, and regularity results of TFNSE (1.2) of order $\alpha \in (0,1)$ in $\mathbb{R}^d$, $d \geq 2$. It is worth mentioning that the decay rate and regularity property depend on the order α and it may provide new insights on the whole scenario to the Navier-Stokes existence and smoothness problem.

Stochastic external forcing is often necessary, particularly in the study of turbulent fluid flows, to probe the nonlinear dynamics. In general, the motivation of the addition of stochastic forcing in the NSE (or TFNSE) is twofold. Firstly, stochastic force is introduced into the NSE (or TFNSE) to obtain an invariant measure for the system. The second justification often invoked is that stochastic force is a reasonable starting point to model the uncertainty in velocity profiles. In this context, it is natural to formulate the stochastic Navier-Stokes equation (SNSE):

$$d\boldsymbol{u} + (-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p)\,dt = \Phi dB_Q(t), \tag{1.3}$$

where Φ is a deterministic function and $B_Q(t)$ is a Q-valued cylindrical Brownian motion under some assumed conditions of Φ and Q. The corresponding Cauchy problems have been extensively studied in the last years [13–21], and this part of the theory is well understood. The independence of increments inherent in Brownian noise is key to all these above studies. For the source of noise, Dong and Xie [22] introduced the Lévy noise rather than the standard Brownian motion and established the existence and uniqueness of the strong and weak solutions for 2D SNSE with Lévy noise on the torus. There is no a prior reason to assume that the stochastic forces are independent over disjoint time intervals, so that Brownian noise and Lévy noise are not appropriate. Motivated by this situation, Fang et al. [23] proposed the following SNSE with an infinite-dimensional fractional Brownian motion

$$d\boldsymbol{u} + (-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p)\,dt = \Phi dB^H(t), \tag{1.4}$$

where $B^H(t)$ is a cylindrical fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$. Hereinabove, they proved the existence and uniqueness of mild solutions to SNSE (1.4) under suitable conditions in L^4. Recently, Li and Huang [24, 25] obtained the well-posedness the mild solutions to stochastic non-Newtonian fluid driven by an fBm in L^2, and established the existence of random attractor. Huang et al. [26, 27] further applied similar idea to establish the well-posedness and dynamics of the stochastic modified Boussinesq approximation equation with an fBm.

To the best of our knowledge, there is no paper which studies the TFNSE driven by an fBm. In this paper, we consider the following 2D time fractional stochastic Navier-Stokes equation (TFSNSE):

$$d\left[I_t^{1-\alpha}(\boldsymbol{u}-\boldsymbol{u}_0)\right]+(-\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla p)\,dt=\Phi dB^H(t), \tag{1.5}$$

where $I_t^{1-\alpha}$ is the $(1-\alpha)$-order Riemann-Liouville fractional integral operator, and Φ is an appropriate function to be specified later.

The purpose of this paper is to investigate the existence and uniqueness of mild solutions to the TFSNSE. Firstly, the TFSNSE is formulated in the abstract form with $\nu = 1$ to simplify the presentation. Then the existence and regularity of the nonlocal stochastic convolution is discussed under some suitable conditions. Last but not least, the well-posedness of the TFSNSE is proved by a modified fixed point theorem.

Compared to the SNSE driven by Brownian motion [13–21], some possible difficulties are encountered due to the presence of the fBm, pointed out by Li and Huang [24]. Motivated by the ideas in [24–27], we adopt the Wiener integrals with respect to fBm and use the notion of mild solution rather than weak solution in this paper. Also, we obtain the well-posedness result in Hilbert space L^2 rather than L^4 compared to the SNSE with an fBm [23]. Let us notice that the analytic semigruop generated by Stokes operator plays a key role in providing some useful estimates in the literature [23–27]. Unfortunately, however, the linear part of the TFSNSE at hand generates a family of Mittag-Leffler operators that do not define a semigruop due to the nonlocal property of fractional differential operator. Combining the theory of fractional calculus and some properties of (generalized) Mittag-Leffler function, we will obtain an upper bound, which is useful to show the regularity of nonlocal stochastic convolution and the existence and uniqueness of mild solution to the TFSNSE.

The paper is organized as follows. Section 2 is preliminaries on basic concepts related to fractional calculus, the formulation of abstract form of the TFSNSE, and stochastic integrals for fBm. In Section 3, two useful estimates are provided and the regularity of a nonlocal stochastic convolution is proved under suitable conditions. Section 4 shows the existence and uniqueness of mild solutions to the TFSNSE.

2 Preliminaries

This section is devoted to briefly present some basic concepts about fractional calculus and special functions, and introduce the definitions of fBm and the Wiener integrals in the separable Hilbert space, which allow us to formulate the abstract form of the TFSNSE and mild solution.

2.1 Fractional calculus

Initially, we recall some basic definitions and results on fractional calculus in Banach space and some special functions.

Throughout this paper, some basic functional spaces are introduced in what follows. Let $T > 0$ and X be a Banach space. Denote by $C([0,T];X)$ the space of the continuous functions from $[0,T]$ to X and $C_b([0,T];X)$ the space of the continuous and bounded functions from $[0,T]$ to X. Also, let $L^p([0,T];X)$ be the Banach space of L^p-integrable functions when $1 \le p < \infty$ and the essential bounded functions if $p = \infty$.

Now we recall the definitions of fractional operators, and refer to the books [28–30] for more details.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of function $f \in L^1([0,T];X)$ is defined by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2.2. The Caputo fractional derivative of order $\alpha \in (0,1)$ of function $f \in C([0,T];X)$ is defined by

$$D_t^\alpha f(t) := \frac{d}{dt}\left[I_t^{1-\alpha}(f(t) - f(0))\right] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds.$$

Let us state some special functions and their properties [31–34].

Definition 2.3. The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, z \in \mathbb{C}.$$

When $\beta = 1$, set $E_\alpha(z) = E_{\alpha,1}(z)$. Also, Mittag-Leffler functions satisfy the following relations:

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \quad \alpha, \beta > 0, z \in \mathbb{C}. \tag{2.1}$$

$$(-1)^m \frac{d^m}{dx^m} E_{\alpha,\beta}(-z) \ge 0, \quad z > 0, 0 < \alpha \le 1, \beta \ge \alpha, m = 0, 1, 2, \ldots. \tag{2.2}$$

$$I_t^\alpha \left(t^{\beta-1} E_{\mu,\beta}(\lambda t^\mu)\right) = t^{\alpha+\beta-1} E_{\mu,\alpha+\beta}(\lambda t^\mu), \quad \alpha, \beta, \mu > 0, \lambda \in \mathbb{C}, t \ge 0. \tag{2.3}$$

$$0 < \frac{1}{1 + t^\alpha \Gamma(1-\alpha)} \le E_\alpha(-t^\alpha) \le \frac{1}{1 + \frac{t^\alpha}{\Gamma(1+\alpha)}} \le 1, \quad 0 < \alpha < 1, t \ge 0. \tag{2.4}$$

Definition 2.4. The Mainardi's function is defined by

$$M_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + 1 - \alpha)}, \quad 0 < \alpha < 1, z \in \mathbb{C}.$$

Moreover, $M_\alpha(z) \ge 0$ for all $z \ge 0$ and satisfies the following equality

$$\int_0^\infty t^r M_\alpha(z) dz = \frac{\Gamma(r+1)}{\Gamma(\alpha r + 1)}, \quad r > -1, 0 < \alpha < 1. \tag{2.5}$$

2.2 Fractional Brownian motion and Wiener integral

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 2.5. Given $H \in (0,1)$, a continuous centered Gaussian stochastic process $\beta^H(t),\ t \in \mathbb{R}$, with the covariance function

$$R_H(t,s) = \mathbb{E}\left[\beta^H(t)\beta^H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad t,s \in \mathbb{R}$$

is called a two-sided one-dimensional fractional Brownian motion (fBm) with Hurst parameter H.

We denote by $\mathcal{S}$ the set of step functions on the finite interval $[0,T]$. Let $\mathcal{K}$ be the Reproducing Kernel Hilbert Space (Cameron-Martin space) defined as the closure of $\mathcal{S}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{K}} = R_H(t,s).$$

Then the map $\mathbf{1}_{[0,t]} \mapsto \beta^H(t)$ extends to an isometry between $\mathcal{K}$ and the $L^2(\Omega)$-closure of the linear span of $\{\beta^H(t) : t \in [0,T]\}$.

Consider the square integrable kernel [35, p.284]

$$K_H(t,s) = c_H\,(t-s)^{H-1/2} + s^{H-1/2} F\left(\frac{t}{s}\right)$$

with $c_H = \sqrt{2H/((1-2H)\beta(1-2H, 1/2+H))}$, and

$$F(z) = c_H\,(1/2 - H) \int_0^{z-1} r^{H-3/2}\left(1 - (1+r)^{H-1/2}\right) dr.$$

Define the adjoint operator K_T^* on a possible subset of $L^2([0,T])$ by

$$\left(K_T^*\phi\right)(s) = K_H(T,s)\phi(s) + \int_s^T (\phi(r) - \phi(s))\,\frac{\partial K_H(r,s)}{\partial r} dr,$$

for $\phi \in \mathcal{K}$. Then K_T^* is an isometry between $\mathcal{K}$ and $L^2([0,T])$, and the Wiener integral with respect to β^H can be represented as

$$\int_0^T \phi(s) d\beta^H(s) = \int_0^T \left(K_T^*\phi\right)(s) dW(s)$$

for all $\phi \in \mathcal{K}$ and $K_T^*\phi \in L^2([0,T])$, where $W(s)$ is the standard Brownian motion. Therefore, we define the Wiener integral

$$\int_0^t \phi(s) d\beta^H(s) = \int_0^t \left(K_t^*\phi\right)(s) dW(s), \quad t \in [0,T].$$

Throughout this paper we limit ourselves to the case $H \in (1/2, 1)$. For a separate Hilbert space $\mathcal{H}$, we can define a Cameron-Martin space

$$\mathscr{H}(0,T;\mathcal{H}) = \overline{\left(L^2(0,T), \langle \cdot,\cdot \rangle_{\mathscr{H}(0,T;\mathcal{H})}\right)}$$

with the twisted product

$$\langle f, g \rangle_{\mathscr{H}(0,T;\mathcal{H})} = H(2H-1) \int_0^T \int_0^T \langle f(s) g(t) \rangle_{\mathcal{H}} |s-t|^{2H-2} ds dt. \tag{2.6}$$

Now we are ready to define the integral with respect to a cylindrical $\mathcal{H}$-valued fBm. Let $\{e_i\}_{i=1}^{\infty}$ be the complete orthonormal basis in $\mathcal{H}$, we define a cylindrical fBm as

$$B^H(t) = \sum_{i=1}^{\infty} e_i \beta_i^H(t), \tag{2.7}$$

where $\beta_i^H(t)$ are one-dimensional fBms mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$.

Next we impose the precise conditions for $\mathcal{O}$ and Φ:

$$\mathcal{O} = [-\pi, \pi] \times [-\pi, \pi], \;\; \Phi = id_{\mathcal{H}}. \tag{2.8}$$

Note that one can not handle the infinite-dimensional problem in a finite dimensional manner due to the present of the genuine cylindrical fBm (2.7) and the fact that the assumed condition (2.8) lack of compactness.

2.3 The abstract form of the TFSNSE

Based on the functional analytic setup [36, 37], we recall notations and functional spaces. Let $\mathcal{U}$ be the space of two-dimensional vector functions $\boldsymbol{u}$ on $\mathcal{O}$ which are infinitely differentiable with compact support strictly contained in $\mathcal{O}$, satisfying $\nabla \cdot \boldsymbol{u} = 0$. Let $W^{\gamma,2}$ be the standard Sobolev space, we thus define the space $\mathcal{H}$ to be the closure of $\mathcal{U}$ in $L^2(\mathcal{O})$, and the space $\mathcal{V}$ to be the closure of $\mathcal{U}$ in $W^{1,2}(\mathcal{O})$. The notation $L^2(\mathcal{O})$, $W_0^{1,2}(\mathcal{O})$, etc. denotes two-vector functions on $\mathcal{O}$ with each coordinate in the scalar versions of $L^2(\mathcal{O})$, $W_0^{1,2}(\mathcal{O})$, etc. In fact, the spaces $\mathcal{H}$ and $\mathcal{V}$ are characterized by

$$\mathcal{H} = \left\{\boldsymbol{u} \in L^2(\mathcal{O}) : \nabla \cdot \boldsymbol{u} = 0, \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\mathcal{O}} = 0\right\},$$

and

$$\mathcal{V} = \left\{\boldsymbol{u} \in W_0^{1,2}(\mathcal{O}) : \nabla \cdot \boldsymbol{u} = 0\right\},$$

where $\boldsymbol{n}$ is the outward normal on $\partial\mathcal{O}$, and

$$W_0^{1,2}(\mathcal{O}) = \left\{\boldsymbol{u} : u_i \in L^2(\mathcal{O}, \mathbb{R}), \nabla u_i \in L^2(\mathcal{O}, \mathbb{R}^2), i = 1, 2, \text{ and } \boldsymbol{n}|_{\partial\mathcal{O}} = 0\right\}.$$

Let $\mathcal{V}'$ be dual of $\mathcal{V}$. We then denote the norm in $\mathcal{H}$ by $|\cdot|$, the inner product in $\mathcal{H}$ by $(\cdot, \cdot)$, the dual pairing between $\mathcal{V}'$ and $\mathcal{V}$ by $\langle \cdot, \cdot \rangle$, and the norm in $W_0^{1,2}(\mathcal{O})$ by $|\cdot|_1$. It follows that $(\mathcal{V}, \mathcal{H}, \mathcal{V}')$ is a Gelfand triple. Denoting by $\mathcal{D}(\boldsymbol{A}) = W^{2,2}(\mathcal{O}) \cap \mathcal{V}$, the well-known Stokes operator $\boldsymbol{A} : \mathcal{D}(\boldsymbol{A}) \to \mathcal{H}$ is given by

$$\boldsymbol{A}\boldsymbol{u} = -\Delta \boldsymbol{u}.$$

By the fact $\mathcal{V} = \mathcal{D}(\boldsymbol{A}^{1/2})$ and Poincaré inequality, we denote the norm in $\mathcal{V}$ by $\|\cdot\|$, and $\|\boldsymbol{u}\| = |\boldsymbol{A}^{1/2}\boldsymbol{u}|$. Since the Stokes operator $\boldsymbol{A}$ is positive, self-adjoint with compact resolvent, there exist eigenvectors $\{e_i\}_{i=1}^{\infty} \subset \mathcal{D}(\boldsymbol{A})$ and eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ such that

$$\boldsymbol{A}e_i = \lambda_i e_i, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \quad \lim_{i\to\infty} \lambda_i = \infty,$$

and $\{e_i\}_{i=1}^{\infty}$ forms a complete orthonormal base for $\mathcal{H}$.

Define the trilinear function $b(\cdot, \cdot, \cdot) : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ by

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

which allows us to define a continuous bilinear operator $\boldsymbol{B} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}'$ as

$$\langle \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w} \rangle = b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}).$$

Note that $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})$. Denote $\boldsymbol{B}(\boldsymbol{u}, \boldsymbol{u})$ by $\boldsymbol{B}(\boldsymbol{u})$ for simplicity.

Assume that $\boldsymbol{u}_0 \in \mathcal{H}$. By applying the Hodge-Leray projection to each term of the TFSNSE (1.5) and invoking the Leary decomposition of $L^2(\mathcal{O})$ into the divergence free and irrotational components, we formally arrive to the abstract evolution equation

$$d\left[I_t^{1-\alpha}\left(\boldsymbol{u}(t) - \boldsymbol{u}_0\right)\right] + \left[\boldsymbol{A}\boldsymbol{u}(t) + \boldsymbol{B}(\boldsymbol{u}(t))\right] dt = dB^H(t), \tag{2.9}$$

under the condition (2.8) and $\nu = 1$ without loss of generality.

In the sequel, we introduce both families of Mittag-Leffler operators based on the analytic semigroup $S(t)$ generated by the Stokes operator $\boldsymbol{A}$:

$$T_\alpha(t) = \int_0^\infty M_\alpha(s) S(st^\alpha) ds,$$

and

$$S_\alpha(t) = \int_0^\infty \alpha s M_\alpha(s) S(st^\alpha) ds.$$

Then we state the notion of mild solution to TFSNSE (2.9).

Definition 2.6. An $\mathcal{H}$-valued process $\boldsymbol{u}(t)$ satisfying

$$\begin{aligned} \boldsymbol{u}(t) &= T_\alpha(t)\,\boldsymbol{u}_0 - \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\,\boldsymbol{B}(\boldsymbol{u}(s)) ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\, dB^H(s) \end{aligned} \tag{2.10}$$

for $t \in [0, T]$ is called a mild solution to (2.9).

We now end this section by providing an estimate on the spectrum of $\boldsymbol{A}$, which is obvious by using [25, Lemma 3.1] and [25, Lemma 3.2].

Lemma 2.7. *Let $\{\lambda_i\}_{i=1}^\infty$ be the eigenvalues of $\boldsymbol{A}$. For $Re(\kappa) > 1/2$*

$$\sum_{i=1}^\infty \lambda_i^{-\kappa} \le \beta_D(2\kappa)\zeta(2\kappa) - \zeta(4\kappa), \tag{2.11}$$

under the condition (2.8), where $\beta_D(\cdot)$ is the Dirichlet beta function and $\zeta(\cdot)$ denotes the Riemann zeta function.

Proof. In fact, it follows from (2.8) and [25, Lemma 3.1] that

$$\sum_{i=1}^\infty \lambda_i^{-\kappa} \le \sum_{i,j=1}^\infty \frac{1}{(i^2 + j^2)^{2\kappa}}.$$

Then we arrive to the required inequality (2.11) by using [25, Lemma 3.2]. □

3 Regularity of the nonlocal stochastic convolution

Consider the nonlocal stochastic convolution

$$z(t) = \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) dB^H(s).$$

Then $z(t)$, if it is well defined in $C([0,T];\mathcal{V})$, is the unique mild solution of the following time fractional linear stochastic evolution equation

$$d\left[I_t^{1-\alpha}\left(z(t)-z_0\right)\right] + \boldsymbol{A}z(t)dt = dB^H(t), \quad z_0 = 0 \in \mathcal{H}.$$

Before proceeding with the existence and regularity of z, we pause to discuss some useful properties of Mittag-Leffler operators.

Lemma 3.1. *Let λ_i and e_i be the ith eigenvalue and eigenvector of $\boldsymbol{A}$. Then*

$$S_\alpha(t)e_i = E_{\alpha,\alpha}(-\lambda_i t^\alpha)e_i, t > 0, 0 < \alpha < 1. \tag{3.1}$$

Proof. It follows from (2.5) and (2.3) that

$$\begin{aligned} S_\alpha(t)e_i &= \int_0^\infty \alpha s M_\alpha(s) S(st^\alpha) e_i ds \\ &= \int_0^\infty \alpha s M_\alpha(s) e^{-\lambda_i s t^\alpha} e_i ds \\ &= \int_0^\infty \alpha s M_\alpha(s) \sum_{k=0}^\infty \frac{(-\lambda_i s t^\alpha)^k}{k!} e_i ds \\ &= \alpha \sum_{k=0}^\infty \frac{(-\lambda_i t^\alpha)^k}{k!} \int_0^\infty s^{k+1} M_\alpha(s) e_i ds \\ &= \alpha \sum_{k=0}^\infty \frac{(-\lambda_i t^\alpha)^k}{k!} \frac{\Gamma(k+2)}{\Gamma(\alpha(k+1)+1)} \\ &= \sum_{k=0}^\infty \frac{(-\lambda_i t^\alpha)^k}{\Gamma(\alpha k + \alpha)} e_i \\ &= E_{\alpha,\alpha}(-\lambda_i t^\alpha) e_i. \end{aligned}$$

□

Lemma 3.2. *Let λ_i be the ith eigenvalue of $\boldsymbol{A}$. Then*

$$E_{\alpha,\alpha+\beta}(-\lambda_i t^\alpha) \le \frac{1}{\Gamma(\beta)} \frac{1}{\lambda_i t^\alpha}, \quad t > 0, \alpha > 0, \beta > 0. \tag{3.2}$$

Proof. It follows from (2.1) with $z = -\lambda_i t^\alpha$ that

$$E_{\alpha,\alpha+\beta}(-\lambda_i t^\alpha) = \frac{1}{\lambda_i t^\alpha}\left(\frac{1}{\Gamma(\beta)} - E_{\alpha,\beta}(-\lambda_i t^\alpha)\right).$$

Also, the case $m = 0$ in (2.2) implies that $E_{\alpha,\beta}(-\lambda_i t^\alpha) \ge 0$. So the required inequality is established. □

We are now in position to state and prove the regularity of the nonlocal stochastic convolution.

Theorem 3.3. *If $H \in (1/2, 1)$, $2 - 2H < \alpha < 1$ and condition (2.8) are satisfied, there is a version of the stochastic convolution $\{z(t), t \in [0, T]\}$ with $C([0, T]; \mathcal{H})$ sample paths.*

Proof. It follows from (2.6) and (2.7) that

$$\begin{aligned}
&\mathbb{E}\,|z(t) - z(s)|^2_{\mathcal{H}} \\
&= \mathbb{E}\left|\int_0^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau) dB^H(\tau) - \int_0^s (s-\tau)^{\alpha-1} S_\alpha(s-\tau) dB^H(\tau)\right|^2_{\mathcal{H}} \\
&\leq 2\mathbb{E}\left|\sum_{i=1}^{\infty}\int_s^t (t-\tau)^{\alpha-1} S_\alpha(t-\tau) e_i d\beta_i^H(\tau)\right|^2_{\mathcal{H}} \\
&\quad + 2\mathbb{E}\left|\sum_{i=1}^{\infty}\int_0^s \left[(t-\tau)^{\alpha-1} S_\alpha(t-\tau) - (s-\tau)^{\alpha-1} S_\alpha(s-\tau)\right] e_i d\beta_i^H(\tau)\right|^2_{\mathcal{H}} \\
&= 2\sum_{i=1}^{\infty}\left|(t-s-\cdot)^{\alpha-1} S_\alpha(t-s-\cdot) e_i\right|^2_{\mathscr{H}(0,t-s;\mathcal{H})} \\
&\quad + 2\sum_{i=1}^{\infty}\left|(t-s+\cdot)^{\alpha-1} S_\alpha(t-s+\cdot) e_i - (\cdot)^{\alpha-1} S_\alpha(\cdot) e_i\right|^2_{\mathscr{H}(0,s;\mathcal{H})} \\
&\triangleq 2I_1 + 2I_2.
\end{aligned}$$

For I_1, it follows from (2.3), (2.6), (2.11), (3.1) and (3.2) that

$$\begin{aligned}
I_1 =& H(2H-1)\sum_{i=1}^{\infty}\int_0^{t-s}\int_0^{t-s} \langle (t-s-\tau_1)^{\alpha-1} S_\alpha(t-s-\tau_1) e_i, \\
&(t-s-\tau_2)^{\alpha-1} S_\alpha(t-s-\tau_2) e_i\rangle_{\mathcal{H}} |\tau_1-\tau_2|^{2H-2} d\tau_1 d\tau_2 \\
=& H(2H-1)\sum_{i=1}^{\infty}\int_0^{t-s}\int_0^{t-s} (t-s-\tau_1)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s-\tau_1)^\alpha) \\
&\times (t-s-\tau_2)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s-\tau_2)^\alpha)\, |\tau_1-\tau_2|^{2H-2} d\tau_1 d\tau_2 \\
=& 2H(2H-1)\sum_{i=1}^{\infty}\int_0^{t-s} (t-s-\tau_1)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s-\tau_1)^\alpha) \\
&\times \int_0^{\tau_1} (t-s-\tau_2)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s-\tau_2)^\alpha)\, (\tau_1-\tau_2)^{2H-2} d\tau_2 d\tau_1 \\
=& \Gamma(2H+1)\sum_{i=1}^{\infty}\int_0^{t-s} (t-s-\tau_1)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s-\tau_1)^\alpha) \\
&\times \left(I_{\tau_1}^{2H-1}\left((t-s-\tau_1)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s-\tau_1)^\alpha)\right)\right) d\tau_1 \\
=& \Gamma(2H+1)\sum_{i=1}^{\infty}\int_0^{t-s} (t-s-\tau_1)^{2\alpha+2H-3} E_{\alpha,\alpha}(-\lambda_i (t-s-\tau_1)^\alpha) \\
&\times E_{\alpha,\alpha+2H-1}(-\lambda_i (t-s-\tau_1)^\alpha) d\tau_1 \\
\leq& \Gamma(2H+1)\sum_{i=1}^{\infty}\int_0^{t-s} (t-s-\tau_1)^{2\alpha+2H-3} \frac{E_{\alpha,\alpha}(-\lambda_i (t-s-\tau_1)^\alpha)}{\Gamma(2H-1)\lambda_i (t-s-\tau_1)^\alpha} d\tau_1 \\
=& 2H(2H-1)\sum_{i=1}^{\infty}\lambda_i^{-1}\int_0^{t-s} (t-s-\tau_1)^{\alpha+2H-3} d\tau_1 \\
\leq& 2H(2H-1)\sum_{i,j=1}^{\infty}\frac{1}{(i^2+j^2)^2}\frac{(t-s)^{\alpha+2H-2}}{\alpha+2H-2}
\end{aligned}$$

$$\leq \frac{2H(2H-1)\left(\beta_D(2)\zeta(2)-\zeta(4)\right)}{\alpha+2H-2}(t-s)^{\alpha+2H-2}.$$

Due to the monotonicity in (2.2) and continuity of the $\alpha-$expenential function $t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})$, there exists some constant C_2 such that

$$\left|(t-s+\tau_1)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s+\tau_1)^{\alpha})-\tau_1^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i\tau_1^{\alpha}))\right| \leq C_2(t-s),$$

$$\left|(t-s+\tau_2)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s+\tau_2)^{\alpha})-\tau_2^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i\tau_2^{\alpha}))\right| \leq \tau_2^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i\tau_2^{\alpha})).$$

Thus, for I_2, it follows that

$$\begin{aligned}
I_2 =& \sum_{i=1}^{\infty}\left|(t-s+\cdot)^{\alpha-1}S_{\alpha}(t-s+\cdot)e_i-(\cdot)^{\alpha-1}S_{\alpha}(\cdot)e_i\right|^2_{\mathscr{H}(0,s;\mathcal{H})}\\
=& H(2H-1)\sum_{i=1}^{\infty}\int_0^s\int_0^s\langle(t-s+\tau_1)^{\alpha-1}S_{\alpha}(t-s+\tau_1)e_i-\tau_1^{\alpha-1}S_{\alpha}(\tau_1)e_i,\\
&(t-s+\tau_2)^{\alpha-1}S_{\alpha}(t-s+\tau_2)e_i-\tau_2^{\alpha-1}S_{\alpha}(\tau_2)e_i\rangle_{\mathcal{H}}\left|\tau_1-\tau_2\right|^{2H-2}d\tau_1 d\tau_2\\
\leq& 2H(2H-1)C_2(t-s)\sum_{i=1}^{\infty}\int_0^s\int_0^{\tau_1}\tau_2^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i\tau_2^{\alpha})(\tau_1-\tau_2)^{2H-2}d\tau_2 d\tau_1\\
=& \Gamma(2H+1)C_2(t-s)\sum_{i=1}^{\infty}\int_0^s\left(I_{\tau_1}^{2H-1}\left(\tau_1^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i\tau_1^{\alpha})\right)\right)d\tau_1\\
=& \Gamma(2H+1)C_2(t-s)\sum_{i=1}^{\infty}\int_0^s\tau_1^{\alpha+2H-2}E_{\alpha,\alpha+2H-1}(-\lambda_i\tau_1^{\alpha})d\tau_1\\
\leq& \Gamma(2H+1)C_2(t-s)\sum_{i=1}^{\infty}\int_0^s\tau_1^{\alpha+2H-2}\frac{1}{\Gamma(2H-1)\lambda_i\tau_1^{\alpha}}d\tau_1\\
=& 2H(2H-1)C_2(t-s)\sum_{i=1}^{\infty}\lambda_i^{-1}\int_0^t\tau_1^{2H-2}d\tau_1\\
=& 2H(2H-1)C_2(t-s)\sum_{i,j=1}^{\infty}\frac{1}{(i^2+j^2)^2}\frac{t^{2H-1}}{2H-1}\\
\leq& \frac{2H(2H-1)\left(\beta_D(2)\zeta(2)-\zeta(4)\right)C_2T^{2H-1}}{2H-1}(t-s).
\end{aligned}$$

So, based on the estimates discussed above, there are some positive constants $C>0, \varepsilon>0$ and for all $s,t\in[0,T]$:

$$\mathbb{E}\left|z(t)-z(s)\right|^2_{\mathcal{H}} \leq C\left|t-s\right|^{\varepsilon}.$$

Therefore the existence and regularity of this nonlocal stochastic convolution are ensured by the Kolmogorov's continuity theorem in a separable Banach space [38, Theorem 3.3] and the fact that the increment of $z(t)$ has Gaussian distribution. □

Theorem 3.4. *If the conditions in Theorem 3.3 are satisfied, then*

$$\mathbb{E}\left|z(t)\right|^2_{\mathcal{H}} < \infty.$$

Proof. One can calculate that

$$
\begin{aligned}
\mathbb{E}\,|z(t)|_{\mathcal{H}}^2 =& \mathbb{E}\left|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)dB^H(s)\right|_{\mathcal{H}}^2 \\
=& \sum_{i=1}^{\infty} \mathbb{E}\left|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)e_i d\beta_i^H(s)\right|_{\mathcal{H}}^2 \\
=& 2H(2H-1)\sum_{i=1}^{\infty}\int_0^t (t-\tau_1)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-\tau_1)^\alpha) \\
& \times \int_0^{\tau_1} (t-\tau_2)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-\tau_2)^\alpha)\,(\tau_1-\tau_2)^{2H-2}\,d\tau_2 d\tau_1 \\
=& \Gamma(2H+1)\sum_{i=1}^{\infty}\int_0^t (t-\tau_1)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-\tau_1)^\alpha)\Big(I_{\tau_1}^{2H-1}\Big((t-\tau_1)^{\alpha-1} \\
& \times E_{\alpha,\alpha}(-\lambda_i(t-\tau_1)^\alpha)\Big)\Big)d\tau_1 \\
=& \Gamma(2H+1)\sum_{i=1}^{\infty}\int_0^t (t-\tau_1)^{2\alpha+2H-3}E_{\alpha,\alpha}(-\lambda_i(t-\tau_1)^\alpha) \\
& \times E_{\alpha,\alpha+2H-1}(-\lambda_i(t-\tau_1)^\alpha)d\tau_1 \\
\le& \sum_{i=1}^{\infty}\int_0^t (t-\tau_1)^{2\alpha+2H-3}E_{\alpha,\alpha}(-\lambda_i(t-\tau_1)^\alpha)\frac{\Gamma(2H+1)}{\Gamma(2H-1)\lambda_i(t-\tau_1)^\alpha}d\tau_1 \\
=& 2H(2H-1)\sum_{i=1}^{\infty}\lambda_i^{-1}\int_0^t (t-\tau_1)^{\alpha+2H-3}d\tau_1 \\
\le& 2H(2H-1)\sum_{i,j=1}^{\infty}\frac{1}{(i^2+j^2)^2}\frac{t^{\alpha+2H-2}}{\alpha+2H-2} \\
\le& \frac{2H(2H-1)\,(\beta_D(2)\zeta(2)-\zeta(4))}{\alpha+2H-2}t^{\alpha+2H-2} \\
<& +\infty.
\end{aligned}
$$

Therefore, the proof of Theorem 3.4 is complete. □

4 Existence and uniqueness of mild solutions

Let us first recall a modified fixed point theorem, which will be applied to ensure the existence and uniqueness of solutions.

Proposition 4.1. *[39, Lemma 15.2.6] Let $\mathcal{T}$ be a transformation from a Banach space $\mathcal{E}$ into $\mathcal{E}$, ϕ a element of $\mathcal{E}$ and $M>0$ a positive number. If $\mathcal{T}(0)=0$, $|\phi|_{\mathcal{E}}\le M/2$ and*

$$|\mathcal{T}(x_1)-\mathcal{T}(x_2)|_{\mathcal{E}} \le \frac{1}{2}|x_1-x_2|_{\mathcal{E}} \;\; \textit{for} \;\; |x_1|_{\mathcal{E}}\le M, |x_2|_{\mathcal{E}}\le M,$$

then the equation

$$u=\phi+\mathcal{T}(u),\;\; u\in\mathcal{E}$$

has a unique solution $u\in\mathcal{E}$ satisfying $|u|_{\mathcal{E}}\le M$.

Our object here is to discuss the Cauchy problem of TFSNSE (2.9) in the Banach space $\mathcal{E}=C([0,T];\mathcal{H})\cap L^2(0,T;\mathcal{V})$.

To this end, we construct the following operator

$$\phi(t) = T_\alpha(t)\boldsymbol{u}_0 + \boldsymbol{z}(t), \tag{4.1}$$

and the transformation $\mathcal{T}$

$$\mathcal{T}(\boldsymbol{u}) = -\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\, \boldsymbol{B}(\boldsymbol{u}(s))ds. \tag{4.2}$$

We next present two lemmas to estimate (4.1) and (4.2) for checking the conditions required by Proposition 4.1.

Lemma 4.2. *If all conditions in Theorem 3.3 are satisfied, then $\phi \in \mathcal{E}$ and*

$$|\phi|_{\mathcal{E}} \leq 2|\boldsymbol{u}_0| + |\boldsymbol{z}|_{\mathcal{E}}. \tag{4.3}$$

for all $\boldsymbol{u}_0 \in \mathcal{H}$.

Proof. By the same argument in Lemma 3.1, it it easy to check that

$$T_\alpha(t)e_i = E_\alpha(-\lambda_i t^\alpha).$$

For $0 < s < t < T$, it follows from the bounds (2.4) that

$$\begin{aligned}
|T_\alpha(t)\boldsymbol{u}_0 - T_\alpha(s)\boldsymbol{u}_0|_{\mathcal{V}} =& \left| \sum_{i=1}^{\infty} T_\alpha(t)(\boldsymbol{u}_0, e_i)e_i - \sum_{i=1}^{\infty} T_\alpha(s)(\boldsymbol{u}_0, e_i)e_i \right|_{\mathcal{V}} \\
\leq& |\boldsymbol{u}_0| \sum_{i=1}^{\infty} |T_\alpha(t)e_i - T_\alpha(s)e_i| \\
=& |\boldsymbol{u}_0| \sum_{i=1}^{\infty} |E_\alpha(-\lambda_i t^\alpha) - E_\alpha(-\lambda_i s^\alpha)| \\
\leq& |\boldsymbol{u}_0| \sum_{i=1}^{\infty} \left(\frac{1}{1 + \frac{\lambda_i s^\alpha}{\Gamma(1+\alpha)}} - \frac{1}{1 + \lambda_i t^\alpha \Gamma(1-\alpha)} \right) \\
=& |\boldsymbol{u}_0| \sum_{i=1}^{\infty} \left(\frac{\lambda_i t^\alpha \Gamma(1-\alpha) - \frac{\lambda_i s^\alpha}{\Gamma(1+\alpha)}}{\left(1 + \frac{\lambda_i s^\alpha}{\Gamma(1+\alpha)}\right)(1 + \lambda_i t^\alpha \Gamma(1-\alpha))} \right) \\
\leq& |\boldsymbol{u}_0| \sum_{i=1}^{\infty} \lambda_i^{-1} \frac{\Gamma(1+\alpha)}{(ts)^\alpha} (t^\alpha - s^\alpha) \\
\leq& \frac{\Gamma(1+\alpha)\left(\beta_D(2)\zeta(2) - \zeta(4)\right)|\boldsymbol{u}_0|}{T^{2\alpha}} (t-s)^\alpha,
\end{aligned}$$

which implies that $T_\alpha(t)\boldsymbol{u}_0 \in C([0,T]; \mathcal{V}) \subset \mathcal{E}$.

Together with $\boldsymbol{z} \in \mathcal{E}$ in Theorem 3.3, we arrive to the required inequality

$$|\phi|_{\mathcal{E}} \leq |T_\alpha(t)\boldsymbol{u}_0|_{\mathcal{E}} + |\boldsymbol{z}|_{\mathcal{E}} \leq 2|\boldsymbol{u}_0| + |\boldsymbol{z}|_{\mathcal{E}}.$$

□

Lemma 4.3. *$\mathcal{T} : \mathcal{E} \to \mathcal{E}$ and for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{E}$, there exist positive constants c_1 and c_2 such that*

$$|\mathcal{T}(\boldsymbol{u})|_{\mathcal{E}}^2 \leq c_1 |\boldsymbol{u}|_{\mathcal{E}}^4, \tag{4.4}$$

$$\begin{aligned}
|\mathcal{T}(\boldsymbol{u}) - \mathcal{T}(\boldsymbol{v})|_{\mathcal{E}}^2 \leq& c_2 \left(|\boldsymbol{u}|^2_{C([0,T];\mathcal{H})} |\boldsymbol{u}|^2_{L^2(0,T;\mathcal{V})} + |\boldsymbol{v}|^2_{C([0,T];\mathcal{H})} |\boldsymbol{v}|^2_{L^2(0,T;\mathcal{V})} \right)^{\frac{1}{2}} \\
&\times |\boldsymbol{u} - \boldsymbol{v}|_{\mathcal{E}}^2 .
\end{aligned} \tag{4.5}$$

Proof. It follows from [39, Lemma 15.2.2] that

$$\boldsymbol{B}(\boldsymbol{u}) \in L^2(0,T;\mathcal{V}'), \ \ \forall \boldsymbol{u} \in \mathcal{E}.$$

Then $\mathcal{T}(\boldsymbol{u})$ is a weak solution of the following time fractional nonhomogeneous linear differential equation

$$\begin{aligned} &D_t^\alpha \mathcal{T}(\boldsymbol{u}(t)) + A\mathcal{T}(\boldsymbol{u}(t)) + \boldsymbol{B}(\boldsymbol{u}(t)) = 0, \ \ t \in [0,T], \\ &\mathcal{T}(\boldsymbol{u}_0) = 0. \end{aligned}$$

Taking inner product of $\mathcal{T}$ in (4.11) and using [40, Lemma 1], we obtain

$$\frac{1}{2} D_t^\alpha \left|\mathcal{T}(\boldsymbol{u}(t))\right|^2 + \left|\mathcal{T}(\boldsymbol{u}(t))\right|_{\mathcal{V}}^2 + \langle \boldsymbol{B}(\boldsymbol{u}(t)), \mathcal{T}(\boldsymbol{u}(t))\rangle \le 0. \tag{4.6}$$

In addition, it follows that

$$-\langle \boldsymbol{B}(\boldsymbol{u}(t)), \mathcal{T}(\boldsymbol{u}(t))\rangle \le \left|\boldsymbol{B}(\boldsymbol{u}(t))\right|_{\mathcal{V}'} \left|\mathcal{T}(\boldsymbol{u}(t))\right|_{\mathcal{V}} \le \frac{1}{2}\left|\boldsymbol{B}(\boldsymbol{u}(t))\right|_{\mathcal{V}'}^2 + \frac{1}{2}\left|\mathcal{T}(\boldsymbol{u}(t))\right|_{\mathcal{V}}^2. \tag{4.7}$$

Substituting (4.7) into (4.6) gives that

$$D_t^\alpha \left|\mathcal{T}(\boldsymbol{u}(t))\right|^2 + \left|\mathcal{T}(\boldsymbol{u}(t))\right|_{\mathcal{V}}^2 \le \left|\boldsymbol{B}(\boldsymbol{u}(t))\right|_{\mathcal{V}'}^2. \tag{4.8}$$

Taking fractional integral I_t^α of both sides in (4.8), it follows that

$$\left|\mathcal{T}(\boldsymbol{u}(t))\right|^2 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\mathcal{T}(\boldsymbol{u}(s))\right|_{\mathcal{V}}^2 ds \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\boldsymbol{B}(\boldsymbol{u}(s))\right|_{\mathcal{V}'}^2 ds.$$

By using [29, Lemma 2.1], there exist positive constants $\tilde{c}_1$ and $\hat{c}_1$ such that

$$\frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\left|\mathcal{T}(\boldsymbol{u}(t))\right|_{\mathcal{V}}^2 dt \le \tilde{c}_1 \int_0^T \left|\mathcal{T}(\boldsymbol{u}(t))\right|_{\mathcal{V}}^2 dt,$$

and

$$\begin{aligned} \frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}\left|\boldsymbol{B}(\boldsymbol{u}(t))\right|_{\mathcal{V}'}^2 dt \le& \tilde{c}_1 \int_0^T \left|\boldsymbol{B}(\boldsymbol{u}(t))\right|_{\mathcal{V}'}^2 dt \\ \le& \tilde{c}_1\hat{c}_1 \int_0^T \left|\boldsymbol{u}(t)\right|^2 \left|\boldsymbol{u}(t)\right|_{\mathcal{V}}^2 dt \\ \le& \tilde{c}_1\hat{c}_1 \left|\boldsymbol{u}\right|_{C([0,T];\mathcal{H})}^2 \int_0^T \left|\boldsymbol{u}(t)\right|_{\mathcal{V}}^2 dt \\ \le& \frac{\tilde{c}_1\hat{c}_1}{2}\left(\left|\boldsymbol{u}\right|_{C([0,T];\mathcal{H})}^4 + \left|\boldsymbol{u}\right|_{L^2(0,T;\mathcal{V})}^4\right) \\ \le& \frac{\tilde{c}_1\hat{c}_1}{2}\left|\boldsymbol{u}\right|_{\mathcal{E}}^4. \end{aligned}$$

Let $c_1 = \hat{c}_1 \max\{\tilde{c}_1, 1\}$, it follows that

$$\left|\mathcal{T}(\boldsymbol{u})\right|_{\mathcal{E}}^2 \le 2\left(\left|\mathcal{T}(\boldsymbol{u})\right|_{C([0,T];\mathcal{H})}^2 + \left|\mathcal{T}(\boldsymbol{u})\right|_{L^2(0,T;\mathcal{V})}^2\right) \le c_1 \left|\boldsymbol{u}\right|_{\mathcal{E}}^4.$$

Next the inequality (4.5) is established. Given $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{E}$, set

$$\boldsymbol{\omega} = \mathcal{T}(\boldsymbol{u}) - \mathcal{T}(\boldsymbol{v}).$$

Then $\boldsymbol{\omega}$ is a weak solution of the following time fractional nonhomogeneous linear differential equation

$$\begin{aligned}&D_t^\alpha \boldsymbol{\omega}(t) + \boldsymbol{A}\boldsymbol{\omega}(t) + (\boldsymbol{B}(\boldsymbol{u}(t)) - \boldsymbol{B}(\boldsymbol{v}(t))) = 0, \ \ t \in [0, T],\\ &\boldsymbol{\omega}(0) = 0.\end{aligned}$$

Similar to the above argument in (4.4) it follows that

$$\begin{aligned}|\boldsymbol{\omega}(t)|^2 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\boldsymbol{\omega}(s)|_{\mathcal{V}}^2 \, ds\\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\boldsymbol{B}(\boldsymbol{v}(s)) - \boldsymbol{B}(\boldsymbol{u}(s))|_{\mathcal{V}'}^2 \, ds,\end{aligned}$$

where

$$\frac{1}{\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} |\boldsymbol{\omega}(t)|_{\mathcal{V}}^2 \, dt \le \tilde{c}_1 \int_0^T |\boldsymbol{\omega}(t)|_{\mathcal{V}}^2 \, dt,$$

and

$$\begin{aligned}\frac{1}{\Gamma(\alpha)} &\int_0^T (T-t)^{\alpha-1} |\boldsymbol{B}(\boldsymbol{u}(t)) - \boldsymbol{B}(\boldsymbol{v}(t))|_{\mathcal{V}'}^2 \, dt\\ &\le \tilde{c}_1 \int_0^T |\boldsymbol{B}(\boldsymbol{u}(t)) - \boldsymbol{B}(\boldsymbol{v}(t))|_{\mathcal{V}'}^2 \, dt.\end{aligned}$$

Since the trilinear function $b(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$ satisfies the inequality

$$b(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) \le \tilde{c}_2 \left(|\boldsymbol{u}_1|^{\frac{1}{2}} |\boldsymbol{u}_1|_{\mathcal{V}}^{\frac{1}{2}} |\boldsymbol{u}_3|_{\mathcal{V}} |\boldsymbol{u}_2|^{\frac{1}{2}} |\boldsymbol{u}_2|_{\mathcal{V}}^{\frac{1}{2}} \right),$$

where $\tilde{c}_2 > 0$ is some constant. Thus for all $\varphi \in \mathcal{V}$, it follows that

$$\begin{aligned}|\langle \boldsymbol{B}(\boldsymbol{u}) - \boldsymbol{B}(\boldsymbol{v}), \varphi \rangle| &= |b(\boldsymbol{u}, \boldsymbol{u}, \varphi) - b(\boldsymbol{v}, \boldsymbol{v}, \varphi)|\\ &= |b(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u}, \varphi) + b(\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}, \varphi)|\\ &\le |b(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u}, \varphi)| + |b(\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}, \varphi)|\\ &\le \tilde{c}_2 \left(|\boldsymbol{u}|^{\frac{1}{2}} |\boldsymbol{u}|_{\mathcal{V}}^{\frac{1}{2}} + |\boldsymbol{v}|^{\frac{1}{2}} |\boldsymbol{v}|_{\mathcal{V}}^{\frac{1}{2}} \right) |\boldsymbol{u} - \boldsymbol{v}|^{\frac{1}{2}} \, |\boldsymbol{u} - \boldsymbol{v}|_{\mathcal{V}}^{\frac{1}{2}} \, |\varphi|_{\mathcal{V}} \, .\end{aligned}$$

Then we further have

$$|\boldsymbol{B}(\boldsymbol{u}(t)) - \boldsymbol{B}(\boldsymbol{v}(t))|_{\mathcal{V}'} \le \tilde{c}_2 \left(|\boldsymbol{u}|^{\frac{1}{2}} |\boldsymbol{u}|_{\mathcal{V}}^{\frac{1}{2}} + |\boldsymbol{v}|^{\frac{1}{2}} |\boldsymbol{v}|_{\mathcal{V}}^{\frac{1}{2}} \right) |\boldsymbol{u} - \boldsymbol{v}|^{\frac{1}{2}} \, |\boldsymbol{u} - \boldsymbol{v}|_{\mathcal{V}}^{\frac{1}{2}} \, .$$

Let $c_2 = 2\tilde{c}_2^2 \max\{\tilde{c}_1, 1\}$, it follows that

$$\begin{aligned}
|\omega|_{\mathcal{E}}^2 \le & 2\left(\sup_{t\in[0,T]} |\omega(t)|^2 + \int_0^T |\omega(t)|_{\mathcal{V}}^2\, dt\right) \\
\le & c_2 \int_0^T \left(|u(t)|^{\frac{1}{2}}|u(t)|_{\mathcal{V}}^{\frac{1}{2}} + |v(t)|^{\frac{1}{2}}|v(t)|_{\mathcal{V}}^{\frac{1}{2}}\right)^2 |u(t)-v(t)|\, |u-v(t)|_{\mathcal{V}}\, dt \\
\le & \frac{c_2}{2}\left(\int_0^T \left(|u(t)|^{\frac{1}{2}}|u(t)|_{\mathcal{V}}^{\frac{1}{2}} + |v(t)|^{\frac{1}{2}}|v(t)|_{\mathcal{V}}^{\frac{1}{2}}\right)^4 |u(t)-v(t)|^2\, dt \right.\\
& \left. + \int_0^T |u-v(t)|_{\mathcal{V}}^2\, dt\right) \\
\le & \frac{c_2}{2}\left(\int_0^T \left(|u(t)|^{\frac{1}{2}}|u(t)|_{\mathcal{V}}^{\frac{1}{2}} + |v(t)|^{\frac{1}{2}}|v(t)|_{\mathcal{V}}^{\frac{1}{2}}\right)^4 |u(t)-v(t)|^2\, dt \right.\\
& \left. + \int_0^T |u-v(t)|_{\mathcal{V}}^2\, dt\right) \\
\le & \frac{c_2}{2}\left(|u(t)-v(t)|_{C([0,T];\mathcal{H})}^2 \int_0^T \left(|u(t)|^{\frac{1}{2}}|u(t)|_{\mathcal{V}}^{\frac{1}{2}} + |v(t)|^{\frac{1}{2}}|v(t)|_{\mathcal{V}}^{\frac{1}{2}}\right)^4 dt \right.\\
& \left. + |u-v(t)|_{L^2(0,T;\mathcal{V})}^2\right) \\
\le & \frac{c_2}{2}\left(|u(t)-v(t)|_{C([0,T];\mathcal{H})}^2 \int_0^T (|u(t)|^2|u(t)|_{\mathcal{V}}^2 + |v(t)|^2|v(t)|_{\mathcal{V}}^2)\, dt \right.\\
& \left. + |u-v(t)|_{L^2(0,T;\mathcal{V})}^2\right) \\
\le & c_2\left(|u|_{C([0,T];\mathcal{H})}^2 |u|_{L^2(0,T;\mathcal{V})}^2 + |v|_{C([0,T];\mathcal{H})}^2 |v|_{L^2(0,T;\mathcal{V})}^2\right)^{\frac{1}{2}} \\
& \times \left(|u(t)-v(t)|_{C([0,T];\mathcal{H})}^2 + |u-v(t)|_{L^2(0,T;\mathcal{V})}^2\right) \\
\le & c_2\left(|u|_{C([0,T];\mathcal{H})}^2 |u|_{L^2(0,T;\mathcal{V})}^2 + |v|_{C([0,T];\mathcal{H})}^2 |v|_{L^2(0,T;\mathcal{V})}^2\right)^{\frac{1}{2}} |u-v(t)|_{\mathcal{E}}^2 .
\end{aligned}$$

□

With these estimates at hand, we are now in a position to state the local existence and uniqueness result to TFSNSE (2.9).

Theorem 4.4. *If all conditions in Theorem 3.3 are satisfied, then there exists $T_0 > 0$ such that TFSNSE (2.9) has a unique mild solution $\{u(t), t \in [0, T_0]\}$ in $C([0, T_0]; \mathcal{H}) \cap L^2(0, T_0; \mathcal{V})$ for all $u_0 \in \mathcal{H}$.*

Proof. Let $M = 4|u_0| + 2|z|_{\mathcal{E}}$. Then it follows from Lemma 4.3 that

$$\begin{aligned}
& |\mathcal{T}(u) - \mathcal{T}(v)|_{\mathcal{E}} \\
& \le c_2\left(|u|_{C([0,T];\mathcal{H})}^2 |u|_{L^2(0,T;\mathcal{V})}^2 + |v|_{C([0,T];\mathcal{H})}^2 |v|_{L^2(0,T;\mathcal{V})}^2\right)^{\frac{1}{4}} |u-v|_{\mathcal{E}} \\
& \le \sqrt{c_2 M}\left(|u|_{L^2(0,T;\mathcal{V})}^2 + |v|_{L^2(0,T;\mathcal{V})}^2\right)^{\frac{1}{4}} |u-v|_{\mathcal{E}} .
\end{aligned} \tag{4.9}$$

By the absolute continuity property of Bochner integral, one can choose T_0 in such a way that

$$\left(|u|_{L^2(0,T_0;\mathcal{V})}^2 + |v|_{L^2(0,T_0;\mathcal{V})}^2\right)^{\frac{1}{4}} \le \left(\sqrt{2}c_2 M\right)^{-\frac{1}{2}} . \tag{4.10}$$

Let us denote that $\mathcal{E}_{T_0} = C([0,T_0];\mathcal{H}) \cap L^2(0,T_0;\mathcal{V})$. Together with (4.9) and (4.10), it yields

$$|\mathcal{T}(\boldsymbol{u}) - \mathcal{T}(\boldsymbol{v})|_{\mathcal{E}_{T_0}} \le \frac{1}{2} |\boldsymbol{u} - \boldsymbol{v}|_{\mathcal{E}_{T_0}}.$$

On the other hand, it follows from (4.3) that

$$|\phi|_{\mathcal{E}_{T_0}} \le \frac{M}{2}.$$

Hence the required result is an immediate consequence by Proposition 4.1. □

The remainder of this section is devoted to showing the existence of the global mild solution for TFSNSE (2.9).

Based on Theorem 4.4, we assume that $\boldsymbol{u}$ is the local mild solution of TFSNSE (2.9) over the interval $[0,T_0]$ and let $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{z}$. Thus it follows from (2.10) that

$$\boldsymbol{v}(t) = T_\alpha(t)\,\boldsymbol{u}_0 - \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\,\boldsymbol{B}(\boldsymbol{v}(s)+\boldsymbol{z}(s))ds,$$

which is the weak solution of the following equation

$$\begin{aligned} &D_t^\alpha \boldsymbol{v}(t) + \boldsymbol{A}\boldsymbol{v}(t) + \boldsymbol{B}(\boldsymbol{v}(t)+\boldsymbol{z}(t)) = 0, \quad t \in [0,T], \\ &\boldsymbol{v}(0) = \boldsymbol{u}_0. \end{aligned} \tag{4.11}$$

Motivated by the idea in [39, Chapter 15], the following a priori estimate is established to describe a bound of $\boldsymbol{v}$ in the space $\mathcal{E}$.

Lemma 4.5. *Assume that $\boldsymbol{v}$ is the solution of (4.11) over $[0,T]$, it follows that*

$$\begin{aligned} |\boldsymbol{v}(t)|^2 &\le |\boldsymbol{v}(0)|^2 E_\alpha\left(c_3 I_t^\alpha |\boldsymbol{z}(t)|_1^2 t^\alpha\right) + I_t^\alpha \chi(t) \\ &\quad + \int_0^t \left[\sum_{k=1}^\infty \frac{\left(c_3 I_t^\alpha |\boldsymbol{z}(t)|_1^2\right)^k}{\Gamma(\alpha k)} (t-s)^{\alpha k-1} I_t^\alpha \chi(s)\right] ds, \end{aligned} \tag{4.12}$$

$$I_T^\alpha |\boldsymbol{v}(T)|_{\mathcal{V}}^2 \le c_4 |\boldsymbol{u}_0|^2 + c_3 c_4 \sup_{t\in[0,T]} |\boldsymbol{v}(t)|^2 I_T^\alpha |\boldsymbol{z}(T)|_1^2 + c_4 I_T^\alpha \chi(T), \tag{4.13}$$

where c_3 and c_4 are positive constants depending on λ_1 and $\mathcal{O}$, and χ is an integrable function depending on $\boldsymbol{z}$.

Proof. Multiplying (4.11) by $\boldsymbol{v}(t)$ and then integrating it over $\mathcal{O}$ in virtue of [40, Lemma 1], we obtain

$$\frac{1}{2} D_t^\alpha |\boldsymbol{v}(t)|^2 + |\boldsymbol{v}(t)|_{\mathcal{V}}^2 + \langle \boldsymbol{B}(\boldsymbol{v}(t)+\boldsymbol{z}(t)), \boldsymbol{v}(t)\rangle = 0.$$

Based on the orthogonality property of the trilinear function $b(\cdot,\cdot,\cdot)$, for every $\gamma > 0$, there exists a constant $\tilde{c}_3 > 0$ such that

$$\begin{aligned} &-\langle \boldsymbol{B}(\boldsymbol{v}(t)+\boldsymbol{z}(t)), \boldsymbol{v}(t)\rangle \\ &= -b(\boldsymbol{v}(t)+\boldsymbol{z}(t), \boldsymbol{v}(t)+\boldsymbol{z}(t), \boldsymbol{v}(t)) \\ &\le |b(\boldsymbol{v}(t)+\boldsymbol{z}(t), \boldsymbol{z}(t), \boldsymbol{v}(t)+\boldsymbol{z}(t)| \\ &\le \tilde{c}_3 |\boldsymbol{v}(t)+\boldsymbol{z}(t)| \cdot |\boldsymbol{z}(t)|_1 \cdot |\boldsymbol{v}(t)+\boldsymbol{z}(t)|_1 \\ &\le \frac{\tilde{c}_3}{2\gamma} |\boldsymbol{v}(t)+\boldsymbol{z}(t)|^2 \cdot |\boldsymbol{z}(t)|_1^2 + \frac{\tilde{c}_3\gamma}{2} |\boldsymbol{v}(t)+\boldsymbol{z}(t)|_1^2 \\ &\le \frac{\tilde{c}_3}{\gamma} |\boldsymbol{v}(t)|^2 \cdot |\boldsymbol{z}(t)|_1^2 + \frac{\tilde{c}_3}{\gamma} |\boldsymbol{z}(t)|^2 \cdot |\boldsymbol{z}(t)|_1^2 + \tilde{c}_3\gamma |\boldsymbol{v}(t)|_1^2 + \tilde{c}_3\gamma |\boldsymbol{z}(t)|_1^2. \end{aligned}$$

Then it follows that

$$\begin{aligned}&\frac{1}{2}D_t^\alpha |\boldsymbol{v}(t)|^2+\frac{\lambda_1}{2}|\boldsymbol{v}(t)|^2+\frac{1}{2}|\boldsymbol{v}(t)|_{\mathcal{V}}^2\\&\quad\le\frac{\tilde{c}_3}{\gamma}|\boldsymbol{v}(t)|^2\cdot|\boldsymbol{z}(t)|_1^2+\frac{\tilde{c}_3}{\gamma}|\boldsymbol{z}(t)|^2\cdot|\boldsymbol{z}(t)|_1^2+\tilde{c}_3\gamma|\boldsymbol{v}(t)|_1^2+\tilde{c}_3\gamma|\boldsymbol{z}(t)|_1^2,\end{aligned}$$

where λ_1 is the first eigenvalue of the Stokes operator $\boldsymbol{A}$.

Letting $\chi(t)=\frac{2\tilde{c}_3}{\gamma}|\boldsymbol{z}(t)|^2\cdot|\boldsymbol{z}(t)|_1^2+2\tilde{c}_3\gamma|\boldsymbol{z}(t)|_1^2$, we further have

$$D_t^\alpha |\boldsymbol{v}(t)|^2+\left(\lambda_1-\frac{2\tilde{c}_3}{\gamma}|\boldsymbol{z}(t)|_1^2\right)|\boldsymbol{v}(t)|^2+\left(1-\frac{2\tilde{c}_3\gamma}{\sqrt{\lambda_1}}\right)|\boldsymbol{v}(t)|_{\mathcal{V}}^2\le\chi(t). \tag{4.14}$$

Since γ was arbitrary, one can choose it in such a way that $2\tilde{c}_3\gamma/\sqrt{\lambda_1}<1$. Then (4.14) becomes

$$D_t^\alpha |\boldsymbol{v}(t)|^2+\left(\lambda_1-\frac{2\tilde{c}_3}{\gamma}|\boldsymbol{z}(t)|_1^2\right)|\boldsymbol{v}(t)|^2\le\chi(t).$$

Let $c_3=2\tilde{c}_3/\gamma$. Then it follows that

$$D_t^\alpha |\boldsymbol{v}(t)|^2\le c_3|\boldsymbol{z}(t)|_1^2\,|\boldsymbol{v}(t)|^2+\chi(t).$$

Moreover, it follows that

$$\begin{aligned}|\boldsymbol{v}(t)|^2&\le\left(|\boldsymbol{v}(0)|^2+I_t^\alpha\chi(t)\right)+c_3I_t^\alpha|\boldsymbol{z}(t)|_1^2I_t^\alpha|\boldsymbol{v}(t)|^2\\&\le\left(|\boldsymbol{v}(0)|^2+I_t^\alpha\chi(t)\right)+\frac{c_3I_t^\alpha|\boldsymbol{z}(t)|_1^2}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|\boldsymbol{v}(s)|^2ds.\end{aligned}$$

Applying the generalized Gronwall inequality [41, Theorem 1], we have

$$\begin{aligned}|\boldsymbol{v}(t)|^2&\le\left(|\boldsymbol{v}(0)|^2+I_t^\alpha\chi(t)\right)\\&\quad+\int_0^t\left[\sum_{k=1}^{\infty}\frac{\left(c_3I_t^\alpha|\boldsymbol{z}(t)|_1^2\right)^k}{\Gamma(\alpha k)}(t-s)^{\alpha k-1}\left(|\boldsymbol{v}(0)|^2+I_t^\alpha\chi(s)\right)\right]ds\\&\le|\boldsymbol{v}(0)|^2E_\alpha\left(c_3I_t^\alpha|\boldsymbol{z}(t)|_1^2t^\alpha\right)\\&\quad+I_t^\alpha\chi(t)+\int_0^t\left[\sum_{k=1}^{\infty}\frac{\left(c_3I_t^\alpha|\boldsymbol{z}(t)|_1^2\right)^k}{\Gamma(\alpha k)}(t-s)^{\alpha k-1}I_t^\alpha\chi(s)\right]ds.\end{aligned}$$

Therefore the inequality (4.12) is sure, so it is sufficient to check (4.13). In fact, taking I_t^α of (4.12) over $[0,T]$, we get

$$|\boldsymbol{v}(T)|^2-|\boldsymbol{v}(0)|^2+\left(1-\frac{2\tilde{c}_3\gamma}{\sqrt{\lambda_1}}\right)I_T^\alpha|\boldsymbol{v}(T)|_{\mathcal{V}}^2\le c_3I_T^\alpha\left(|\boldsymbol{z}(T)|_1^2|\boldsymbol{v}(T)|^2\right)+I_T^\alpha\chi(T).$$

Let $c_4=(1-2\tilde{c}_3\gamma{\lambda_1}^{-1/2})^{-1}$, where $\gamma>0$ is small enough such that $c_4>0$. Then we arrive at

$$I_T^\alpha|\boldsymbol{v}(T)|_{\mathcal{V}}^2\le c_4|\boldsymbol{u}_0|^2+c_3c_4\sup_{t\in[0,T]}|\boldsymbol{v}(t)|^2I_T^\alpha|\boldsymbol{z}(T)|_1^2+c_4I_T^\alpha\chi(T).$$

Therefore the proof of Lemma 4.5 is complete. □

Combining all of these results, we can now prove the concrete result of this paper.

Theorem 4.6. *If all conditions in Theorem 3.3 are satisfied, then TFSNSE (2.9) has a unique mild solution $\{\boldsymbol{u}(t),t\in[0,T]\}$ in $C([0,T];\mathcal{H})\cap L^2(0,T;\mathcal{V})$ for all $\boldsymbol{u}_0\in\mathcal{H}$.*

Proof. It follows immediately from the existence and uniqueness of local mild solution by Theorem 4.4, and the a priori estimate of Lemma 4.5. □

5 Conclusion

We have analyzed the global well-posedness of the time fractional stochastic Navier-Stokes equation (TFSNSE) with a genuine cylindrical fractional Brownian motion (fBm). Since the mild solution is represented by a family of Mittag-Leffler operators rather than the analytic semigroup generated by the Stokes operator, we have established a upper bound of Mittag-Leffler operators to overcome this difficulty. Moreover, we have shown that both of the Hurst parameter and the fractional derivative can influence the regularity of the nonlocal stochastic convolution as well as the mild solution.

Note that we restrict ourselves to the long memory case since the computation for the regularity is different for $H \in (1/2, 1)$ and $H \in (0, 1/2)$. It is highlighted that we need more requirements to deal with the lower regularity case $H \in (0, 1/2)$. In particular, it is possible to represent the fBm in terms of the standard cylindrical Brownian motion by using the fractional Riemann-Liouville integral when $1/4 < H < 1/2$. Thus one can obtain the regularity under some suitable conditions and further discuss the well-posedness and dynamics in this case. As for the case $H \in (0, 1/4)$, there is no result on the computation for the regularity the corresponding support in rough path topology is not available. Moreover, it is worth to observe that one should expect, at least formally, that solutions can generate a random dynamical systems due to the pathwise Wiener integral. This interesting issue is out of the scope of this paper, however it will be the subject of a forthcoming work.

6 Declarations

Competing Interests

The author(s) declare that they have no competing interests.

Ethical Approval

Not applicable.

Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

Availability Data and Materials

Not applicable.

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